96. *Calculus in Ranked Vector Spaces. VI*

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3.4. Differentiable mappings into a direct product.

(3.4.1) **Proposition.** *Let $F_1, F_2, \cdots, F_m$; $E_1, E_2, \cdots, E_m$ be a family of separated ranked vector spaces and $f_1: F_1 \to E_1$, $f_2: F_2 \to E_2, \cdots, f_m: F_m \to E_m$ a family of mappings. Let $\times f_i: \times F_i \to \times E_i$ be a mapping from $\times F_i$ into $\times E_i$ defined by,*

$$(\times f_i)(x) = (f_1(x_1), f_2(x_2), \cdots, f_m(x_m))$$

for any element $x = (x_1, x_2, \cdots, x_m) \in \times F_i$. Then $\times f_i: \times F_i \to \times E_i$ is differentiable at the point $a = (a_1, a_2, \cdots, a_m) \in \times F_i$ if and only if for each i ($i = 1, 2, \cdots, m$) $f_i: F_i \to E_i$ is differentiable at the point $a_i \in F_i$, and then

$$(\times f_i)'(a) = \times f_i'(a_i).$$

Proof. (a) Suppose that $\times f_i: \times F_i \to \times E_i$ is differentiable at the point $a = (a_1, a_2, \cdots, a_m) \in \times F_i$, i.e., there exists a map $\times l_i \in L(\times F_i; \times E_i)$ such that the map $\times r_i: \times F_i \to \times E_i$ defined by

$$(\times f_i)(a+h) = (\times f_i)(a) + (\times l_i)(h) + (\times r_i)(h)$$

is a remainder, where $h = (h_1, h_2, \cdots, h_m) \in \times F_i$.

$$\therefore \quad f_i(a_i + h_i) = f_i(a_i) + l_i(h_i) + r_i(h_i), \qquad i = 1, 2, \cdots, m.$$

We shall show that it follows from $\times l_i \in L(\times F_i; \times E_i)$ that for each i ($i = 1, 2, \cdots, m$)

$$l_i \in L(F_i; E_i).$$

In fact, by $\times l_i \in L(\times F_i; \times E_i)$,

$$(\times l_i)(h + h') = (\times l_i)(h) + (\times l_i)(h')$$

where $h = (h_1, h_2, \cdots, h_m)$, $h' = (h_1', h_2', \cdots, h_m')$ are arbitrary elements of $\times F_i$. From this we have

$$\begin{aligned} &(l_1(h_1 + h_1'), l_2(h_2 + h_2'), \cdots, l_m(h_m + h_m')) \\ &= (l_1(h_1), l_2(h_2), \cdots, l_m(h_m)) + (l_1(h_1'), l_2(h_2'), \cdots, l_m(h_m')) \\ &= (l_1(h_1) + l_1(h_1'), l_2(h_2) + l_2(h_2'), \cdots, l_m(h_m) + l_m(h_m')). \end{aligned}$$

$$\therefore \quad l_i(h_i + h_i') = l_i(h_i) + l_i(h_i'), \qquad i = 1, 2, \cdots, m.$$

That is, $l_1, l_2, \cdots, l_m$ are linear.

By $\times l_i \in L(\times F_i; \times E_i)$, $\times l_i$ is continuous, and therefore it is obvious that l_i is continuous.

$$\therefore \quad l_i \in L(F_i; E_i), \qquad i = 1, 2, \cdots, m.$$

We shall next show that $\times r_i \in R(\times F_i; \times E_i)$ implies $r_i \in R(F_i; E_i)$, $i = 1, 2, \cdots, m$.

Let $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ be a quasi-bounded sequence in $\times F_i$ and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n \to 0$, then $\times r_i \in R(\times F_i ; \times E_i)$ implies

$$\left\{\lim \frac{(\times r_i)(\lambda_n z_n)}{\lambda_n}\right\} \ni 0$$

$$\therefore \quad \left\{\lim \left(\frac{r_1(\lambda_n x_{n1})}{\lambda_n}, \frac{r_2(\lambda_n x_{n2})}{\lambda_n}, \cdots, \frac{r_m(\lambda_n x_{nm})}{\lambda_n}\right)\right\} \ni 0$$

$$\therefore \quad \left\{\lim \frac{r_1(\lambda_n x_{n1})}{\lambda_n}\right\} \ni 0, \left\{\lim \frac{r_2(\lambda_n x_{n2})}{\lambda_n}\right\} \ni 0, \cdots,$$

$$\left\{\lim \frac{r_m(\lambda_n x_{nm})}{\lambda_n}\right\} \ni 0.$$

$$\therefore \quad r_1 \in R(F_1 ; E_1),\ r_2 \in R(F_2 ; E_2), \cdots, r_m \in R(F_m ; E_m).$$

Therefore $f_i : F_i \to E_i$ is differentiable at the point $a_i \in F_i$, for $i=1, 2, \cdots, m$.

(b) Suppose conversely that for each i $(i=1, 2, \cdots, m)$ $f_i : F_i \to E_i$ is differentiable at the point $a_i \in F_i$, i.e.,

$$f_1(a_1+h_1)=f_1(a_1)+l_1(h_1)+r_1(h_1)$$
$$f_2(a_2+h_2)=f_2(a_2)+l_2(h_2)+r_2(h_2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_m(a_m+h_m)=f_m(a_m)+l_m(h_m)+r_m(h_m)$$

where $l_1 \in L(F_1 ; E_1)$, $l_2 \in L(F_2 ; E_2)$, $\cdots$, $l_m \in L(F_m ; E_m)$; $r_1 \in R(F_1 ; E_1)$, $r_2 \in R(F_2 ; E_2)$, $\cdots$, and $r_m \in R(F_m ; E_m)$. Thus we have

$$(\times f_i)(a+h)=(\times f_i)(a)+(\times l_i)(h)+(\times r_i)(h)$$

where $a=(a_1, a_2, \cdots, a_m)$ and $h=(h_1, h_2, \cdots, h_m) \in \times F_i$.

It only remains to prove that

$$\times l_i \in L(\times F_i ; \times E_i) \quad \text{and} \quad \times r_i \in R(\times F_i ; \times E_i).$$

It is clear that we have

$$\times l_i \in L(\times F_i ; \times E_i).$$

In fact, let $h=(h_1, h_2, \cdots, h_m)$, $h'=(h'_1, h'_2, \cdots, h'_m)$ be arbitrary elements in $\times F_i$, then

$$\begin{aligned}(\times l_i)(h+h')&=(l_1(h_1+h'_1), l_2(h_2+h'_2), \cdots, l_m(h_m+h'_m))\\ &=(l_1(h_1)+l_1(h'_1), l_2(h_2)+l_2(h'_2), \cdots, l_m(h_m)+l_m(h'_m))\\ &=(l_1(h_1), l_2(h_2), \cdots, l_m(h_m))+(l_1(h'_1), l_2(h'_2), \cdots, l_m(h'_m))\\ &=(\times l_i)(h)+(\times l_i)(h')\end{aligned}$$

i.e., $\times l_i : \times F_i \to \times E_i$ is linear.

Since $l_i : F_i \to E_i$ $(i=1, 2, \cdots, m)$ are continuous, $\times l_i : \times F_i \to \times E_i$ is also continuous,

$$\therefore \quad \times l_i \in L(\times F_i ; \times E_i).$$

We shall next show that $\times r_i \in R(\times F_i ; \times E_i)$. For this, let $\{z_n\} =\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ be any quasi-bounded sequence in $\times F_i$ and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ with $\lambda_n \to 0$, then

$$\frac{(\times r_i)(\lambda_n z_n)}{\lambda_n} = \left(\frac{r_1(\lambda_n x_{n1})}{\lambda_n}, \frac{r_2(\lambda_n x_{n2})}{\lambda_n}, \cdots, \frac{r_m(\lambda_n x_{nm})}{\lambda_n}\right).$$

Since $r_1 \in R(F_1; E_1)$, $r_2 \in R(F_2; E_2)$, $\cdots$, $r_m \in R(F_m; E_m)$,

$$\left\{\lim \frac{r_1(\lambda_n x_{n1})}{\lambda_n}\right\} \ni 0, \left\{\lim \frac{r_2(\lambda_n x_{n2})}{\lambda_n}\right\} \ni 0, \cdots, \left\{\lim \frac{r_m(\lambda_n x_{nm})}{\lambda_n}\right\} \ni 0.$$

By (1.5.1) we get

$$\therefore \quad \left\{\lim \frac{(\times r_i)(\lambda_n z_n)}{\lambda_n}\right\} \ni 0$$

$$\therefore \quad \times r_i \in R(\times F_i; \times E_i).$$

Thus $\times f_i : \times F_i \to \times E_i$ is differentiable at the point $a=(a_1, a_2, \cdots, a_m) \in \times F_i$ and

$$(\times f_i)'(a) = \times f_i'(a_i).$$

(3.4.2) **Proposition.** *Let $E, E_1, E_2, \cdots, E_m$ be a family of separated ranked vector spaces and $f_1 : E \to E_1$, $f_2 : E \to E_2$, $\cdots$, $f_m : E \to E_m$ a family of mappings. Let Πf_i be a mapping from E into $\times E_i$ defined by*

$$(\Pi f_i)(x) = (f_1(x), f_2(x), \cdots, f_m(x))$$

for any $x \in E$. Then $\Pi f_i : E \to \times E_i$ is differentiable at a point $a \in E$ if and only if for each i ($i=1, 2, \cdots, m$) $f_i : E \to E_i$ is differentiable at the point $a \in E$, and then

$$(\Pi f_i)'(a) = \Pi f_i'(a).$$

Proof. (a) Suppose that $\Pi f_i : E \to \times E_i$ is differentiable at the point $a \in E$. Let us consider for each k ($k=1, 2, \cdots, m$) a mapping $P_k : \times E_i \to E_k$ defined by

$$P_k z = x_k$$

for any element $z=(x_1, x_2, \cdots, x_m) \in \times E_i$. Then it is clear that $P_k : \times E_i \to E_k$ is linear and continuous, and therefore by (3.2.1) it is differentiable at each point in $\times E_i$.

Since

$$f_k = P_k \cdot \Pi f_i,$$

by the chain rule f_k is differentiable at the point $a \in E$.

(b) Suppose conversely that $f_1 : E \to E_1$, $f_2 : E \to E_2$, $\cdots$, $f_m : E \to E_m$ are differentiable at the point $a \in E$. Then by (3.4.1) the mapping

$$\times f_i : \times E = E^m \to \times E_i$$

is differentiable at the point $(a, a, \cdots, a) \in E^m$.

Now let us consider a mapping $d : E \to E^m$ defined by

$$d(x) = (x, x, \cdots, x) \in E^m$$

for any $x \in E$. It is obvious that $d : E \to E^m$ is linear and continuous. Hence by (3.2.1) it is differentiable and $d'(a)=d$.

Since

$$\Pi f_i = \times f_i \cdot d,$$

by the chain rule Πf_i is differentiable at the point $a \in E$, and

$$(\Pi f_i)'(a) = \Pi f_i'(a).$$

References

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