95. *Calculus in Ranked Vector Spaces.* V

By Masae YAMAGUCHI
Department of Mathematics, University of Hokkaido

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(2.1.8) **Proposition.** *If E_2 is a separated ranked vector space, then the only remainder $r \in R(E_1; E_2)$ which is linear is the zero map.*

Proof. Let x be an arbitrary point of E_1 and consider a sequence $\{x_n\}$ such that $x_n = x$ for $n = 0, 1, 2, \cdots$. Then by (1.7.3) $\{x_n\}$ is a quasi-bounded sequence. Let $\{\lambda_n\}$ be a sequence in $\mathfrak{R}$ with $\lambda_n \to 0$, then it follows from $r \in R(E_1; E_2)$ that

$$\left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0.$$

The linearity of r implies

$$\frac{r(\lambda_n x_n)}{\lambda_n} = \frac{\lambda_n r(x_n)}{\lambda_n} = r(x_n)$$

$$\therefore \quad \{\lim r(x_n)\} \ni 0.$$

On the other hand, using $r(x_n) = r(x)$ for $n = 0, 1, 2, \cdots$ and (1.2.4), we have

$$\{\lim r(x_n)\} \ni r(x).$$

Since E_2 is a separated ranked vector space, by (1.4.3)

$$r(x) = 0.$$

Hence $r: E_1 \to E_2$ is the zero map.

2.2. Differentiability at a point. In order to make use of (2.1.8) we assume henceforth that all spaces $E_1, E_2, \cdots$ are separated.

(2.2.1) **Proposition.** *Let $f: E_1 \to E_2$ be a map between ranked vector spaces E_1, E_2. If there exists a map $l \in L(E_1; E_2)$ such that the map $r: E_1 \to E_2$ defined by*

$$f(a+h) = f(a) + l(h) + r(h)$$

is a remainder, then l is uniquely determined.

Proof. Suppose that there exist two maps $l_1, l_2 \in L(E_1; E_2)$ such that the maps r_1, r_2 defined by

$$f(a+h) = f(a) + l_1(h) + r_1(h),$$
$$f(a+h) = f(a) + l_2(h) + r_2(h)$$

are remainders. Then we have

$$l_1(h) - l_2(h) = r_2(h) - r_1(h).$$

Since by (2.1.4) $R(E_1; E_2)$ is a vector space and by (2.1.5) $L(E_1; E_2)$ is also a vector space,

$$r_2 - r_1 \in R(E_1; E_2) \quad \text{and} \quad r_2 - r_1 \in L(E_1; E_2).$$

Hence it follows, using (2.1.8), that

$$r_2 - r_1 = 0 \qquad \therefore \quad l_1 = l_2$$

which completes the proof.

(2.2.2) **Definition.** If there exists a map $l \in L(E_1; E_2)$ such that the map $r: E_1 \to E_2$ defined by

$$f(a+h) = f(a) + l(h) + r(h)$$

is a remainder, then the map $f: E_1 \to E_2$ is said to be *differentiable at the point* a and the map $l \in L(E_1; E_2)$ which by (2.2.1) is uniquely determined, is then called the *derivative of* f *at the point* a. It will be denoted as follows:

$$l = Df(a) \quad \text{or} \quad l = f'(a).$$

(2.2.3) **Example.** A constant map $K: E_1 \to E_2$ is differentiable at each point $a \in E_1$, and $DK(a) = 0$.

(2.2.4) **Proposition.** *If* $f: E_1 \to E_2$ *is differentiable at a point* a, *then it is continuous at the point* a *in the sense of L-convergence.*

Proof. Let $\{\mathrm{Lim}\, x_n\} \ni a$, i.e.,

$$x_n - a = \lambda_n x'_n, \qquad \text{for } n = 0, 1, 2, \cdots$$

where $\lambda_n \to 0$ in $\mathfrak{R}$ and $\{x'_n\}$ is a quasi-bounded sequence in E_1.

By assumption we have

$$f(a+h) = f(a) + l(h) + r(h)$$

where $l \in L(E_1; E_2)$ and $r \in R(E_1; E_2)$. Hence

$$\begin{aligned} f(x_n) &= f(a + x_n - a) \\ &= f(a) + l(x_n - a) + r(x_n - a) \\ &= f(a) + l(\lambda_n x'_n) + r(\lambda_n x'_n) \end{aligned}$$

$$\therefore \quad f(x_n) - f(a) = l(\lambda_n x'_n) + r(\lambda_n x'_n).$$

Since $l \in L(E_1; E_2)$,

$$= \lambda_n l(x'_n) + r(\lambda_n x'_n)$$

$$\therefore \quad f(x_n) - f(a) = \lambda_n \left(l(x'_n) + \frac{r(\lambda_n x'_n)}{\lambda_n} \right).$$

By $r \in R(E_1; E_2)$

$$\left\{ \lim \frac{r(\lambda_n x'_n)}{\lambda_n} \right\} \ni 0$$

and therefore $\left\{ \dfrac{r(\lambda_n x'_n)}{\lambda_n} \right\}$ is a quasi-bounded sequence.

Thus it follows from (1.7.7), (1.7.5) that

$$\left\{ l(x'_n) + \frac{r(\lambda_n x'_n)}{\lambda_n} \right\}$$

is also a quasi-bounded sequence.

$$\therefore \quad \{\mathrm{Lim}\, f(x_n)\} \ni f(a)$$

which completes the proof.

2.3. The chain rule. (2.3.1) **Theorem.** *Let* E_1, E_2, E_3 *be ranked vector spaces, and suppose that there are two maps given:*

$$f : E_1 \to E_2, \qquad g : E_2 \to E_3.$$

If $f : E_1 \to E_2$ *is differentiable at a point* $a \in E_1$ *and* $g : E_2 \to E_3$ *is differentiable at the point* $b = f(a) \in E_2$, *then* $g \cdot f$ *is differentiable at the point* $a \in E_1$ *and*

$$D(g \cdot f) = Dg(b) \cdot Df(a).$$

Proof. By assumption we have

$$f(a+h) = f(a) + l_1(h) + r_1(h),$$
$$g(b+k) = g(b) + l_2(k) + r_2(k)$$

where $l_1 = Df(a) \in L(E_1; E_2)$, $l_2 = Dg(b) \in L(E_2; E_3)$, $r_1 \in R(E_1; E_2)$, and $r_2 \in R(E_2; E_3)$.

$$g(f(a+h)) = g(b+k)$$

where $k = l_1(h) + r_1(h)$

$$= g(b) + l_2(k) + r_2(k)$$
$$= g(f(a)) + l_2(l_1(h) + r_1(h)) + r_2(l_1(h) + r_1(h))$$
$$= (g \cdot f)(a) + (l_2 \cdot l_1)(h) + (l_2 \cdot r_1)(h) + (r_2 \cdot (l_1 + r_1))(h)$$

since $l_1 \in L(E_1; E_2)$ and $l_2 \in L(E_2; E_3)$,

$$l_2 \cdot l_1 \in L(E_1; E_3).$$

By (2.1.4), (2.1.6), (2.1.7)

$$l_2 \cdot r_1 + r_2 \cdot (l_1 + r_1) \in R(E_1; E_3).$$

Therefore $g \cdot f$ is differentiable at the point $a \in E_1$ and

$$D(g \cdot f) = l_2 \cdot l_1 = Dg(b) \cdot Df(a).$$

§ 3. Examples and special cases. 3.1. The classical case.

(3.1.1) **Proposition.** *If* E_1, E_2 *are normed vector spaces, on which we consider the ranked topology, as in* (1.6.6), *determined by the norm, then the notions of differentiability at a point* $a \in E_1$ *and derivative of a map* $f : E_1 \to E_2$ *coincide with the classical notions in the sense of Fréchet.*

Proof. (a) Suppose that a map $f : E_1 \to E_2$ is differentiable at a point $a \in E_1$ in the sense of Fréchet, i.e., there exists a map $l \in L(E_1; E_2)$ such that the map r defined by $f(a+h) = f(a) + l(h) + r(h)$ has the following property:

$$\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0, \tag{3.1.2}$$

where $f'(a) = l$.

Let $\{x_n\}$ be any quasi-bounded sequence in E_1, and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ with $\lambda_n \to 0$, then by (1.9.1)

$$\|\lambda_n x_n\| \to 0, \qquad \text{for } n \to \infty.$$

Consider

$$\left\| \frac{r(\lambda_n x_n)}{\lambda_n} \right\| = \|x_n\| \frac{\|r(\lambda_n x_n)\|}{\|\lambda_n x_n\|}$$

then it follows, using (3.1.2) and the fact that by (1.9.2) $\{\|x_n\|\}$ is bounded, that

$$\left\| \frac{r(\lambda_n x_n)}{\lambda_n} \right\| \to 0, \qquad \text{for } n\to\infty$$

$$\therefore \quad \left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0.$$

That is, $r: E_1 \to E_2$ is a remainder, and therefore $f: E_1 \to E_2$ is differentiable at a point $a \in E_1$ in the sense of ranked vector space.

(b) Suppose conversely that $f: E_1 \to E_2$ is differentiable at a point $a \in E_1$ in the sense of ranked vector space, i.e., it can be written in the following way:

$$f(a+h) = f(a) + l(h) + r(h)$$

where $l \in L(E_1; E_2)$ and $r \in R(E_1; E_2)$.

Let $\{x_n\}$ be any sequence in E_1 such that

$$\{\lim_n x_n\} \ni 0, \ \text{i.e.,} \ \lim_{n\to\infty} ||x_n|| = 0,$$

and put

$$y_n = \frac{x_n}{||x_n||}, \qquad n = 0, 1, 2, \cdots,$$

then by (1.9.2) $\{y_n\}$ is a quasi-bounded sequence.

$$\frac{||r(x_n)||}{||x_n||} = \frac{||r(||x_n||y_n)||}{||x_n||}.$$

Since $r \in R(E_1; E_2)$,

$$\frac{||r(||x_n||y_n)||}{||x_n||} \to 0, \qquad \text{for } n\to\infty$$

$$\therefore \quad \lim \frac{||r(x_n)||}{||x_n||} = 0$$

$$\therefore \quad \lim_{h\to 0} \frac{||r(h)||}{||h||} = 0.$$

Therefore $f: E_1 \to E_2$ is differentiable at a point $a \in E_1$ in the sense of Fréchet.

It is obvious that the derivative of a map $f: E_1 \to E_2$ coincides with the classical one in the sense of Fréchet.

3.2. Linear and bilinear maps. (3.2.1) **Proposition.** *Let $f: E_1 \to E_2$ be a linear and continuous map between separated ranked vector spaces E_1 and E_2, then it is differentiable at each point $a \in E_1$ and $f'(a) = f$.*

Proof. By assumption we have

$$f(a+h) = f(a) + f(h) + 0$$

where $f \in L(E_1; E_2)$ and $0 \in R(E_1; E_2)$. Hence $f: E_1 \to E_2$ is differentiable at each point $a \in E$ and $f'(a) = f$.

(3.2.2) **Proposition.** *Let $b: E_1 \times E_2 \to E_3$ be a bilinear and continuous map between separated ranked vector spaces $E_1 \times E_2$, E_3. Then*

$$b \in R(E_1 \times E_2; E_3).$$

Proof. (1) It is obvious that one has $b(0)=0$.

(2) Let $\{z_n\}=\{(x_{n1}, x_{n2})\}$ be a quasi-bounded sequence in $E_1\times E_2$ and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n\to 0$.

$$\theta_b(\lambda_n, z_n)=\frac{b(\lambda_n z_n)}{\lambda_n}=\frac{b(\lambda_n x_{n1}, \lambda_n x_{n2})}{\lambda_n}$$
$$=\lambda_n b(x_{n1}, x_{n2})$$
$$=\pm b(\sqrt{|\lambda_n|}x_{n1}, \sqrt{|\lambda_n|}x_{n2})$$

since by (1.7.8) $\{x_{n1}\}$, $\{x_{n2}\}$ are quasi-bounded sequences and $\sqrt{|\lambda_n|}\to 0$ in $\mathfrak{R}$,

$$\{\lim \sqrt{|\lambda_n|}x_{n1}\}\ni 0 \quad\text{and}\quad \{\lim \sqrt{|\lambda_n|}x_{n2}\}\ni 0.$$

It follows, using that $b: E_1\times E_2\to E_3$ is continuous, that

$$\{\lim \theta_b(\lambda_n, z_n)\}\ni 0$$
$$\therefore\quad b\in R(E_1\times E_2; E_3).$$

(3.2.3) **Proposition.** *Let $b: E_1\times E_2\to E_3$ be bilinear and continuous. Then b is differentiable at each point $a=(a_1, a_2)\in E_1\times E_2$ and*

$$b'(a_1, a_2)(h_1, h_2)=b(h_1, a_2)+b(a_1, h_2).$$

Proof. Let $a=(a_1, a_2)$ and $h=(h_1, h_2)$, then

$$b(a+h)=b(a_1+h_1, a_2+h_2)$$
$$=b(a_1, a_2)+b(h_1, a_2)+b(a_1, h_2)+b(h_1, h_2).$$

Put

$$l(h)=b(h_1, a_2)+b(a_1, h_2), \qquad r(h)=b(h),$$

then it is obvious that one has

$$l\in L(E_1\times E_2; E_3),$$

and by (3.2.2)

$$r\in R(E_1\times E_2; E_3).$$

Hence $b: E_1\times E_2\to E_3$ is differentiable at the point $a=(a_1, a_2)$ and

$$b'(a_1, a_2)(h_1, h_2)=b(h_1, a_2)+b(a_1, h_2).$$

3.3. The special case $f:\mathfrak{R}\to E$. (3.3.1) **Proposition.** *If $f:\mathfrak{R}\to E$ is differentiable at a point $\alpha\in\mathfrak{R}$, then for any sequence $\{x_n\}$ in $\mathfrak{R}$ such that $x_n\to 0$,*

$$\left\{\lim \frac{f(\alpha+x_n)-f(\alpha)}{x_n}\right\}\ni f^{\cdot}(\alpha), \tag{3.3.2}$$

where $f^{\cdot}(\alpha)=f'(\alpha)(1)$.

Proof. By assumption we have

$$f(\alpha+h)=f(\alpha)+l(h)+r(h)$$

where $l\in L(\mathfrak{R}; E)$ and $r\in R(\mathfrak{R}; E)$. Thus

$$f(\alpha+x_n)=f(\alpha)+l(x_n)+r(x_n),$$

since $l\in L(\mathfrak{R}; E)$ implies $l(x_n)=l(x_n\cdot 1)=x_n l(1)$,

$$f(\alpha+x_n)-f(\alpha)=x_n l(1)+r(x_n)$$
$$\frac{f(\alpha+x_n)-f(\alpha)}{x_n}-l(1)=\frac{r(x_n)}{x_n}.$$

It follows from $r \in R(\mathfrak{R}; E)$ that

$$\left\{\lim \frac{r(x_n \cdot 1)}{x_n}\right\} \ni 0$$

$$\therefore \quad \left\{\lim \frac{f(\alpha+x_n)-f(\alpha)}{x_n}\right\} \ni l(1).$$

Since we assume that E is a separated ranked vector space, we may write in the following way:

$$\lim_n \frac{f(\alpha+x_n)-f(\alpha)}{x_n} = f^{\cdot}(\alpha)$$

instead of (3.3.2).

(3.3.3) **Proposition.** *Suppose that for any sequence $\{x_n\}$ in $\mathfrak{R}$ with $x_n \to 0$ the following holds at a point $\alpha \in \mathfrak{R}$:*

$$\left\{\lim \frac{f(\alpha+x_n)-f(\alpha)}{x_n}\right\} \ni a$$

where a is an element of a separated ranked vector space E. Then $f : \mathfrak{R} \to E$ is differentiable at the point $\alpha \in R$ and $f'(\alpha)(x) = xa$.

Proof. Let us define a map $r : \mathfrak{R} \to E$ by

$$f(\alpha+x) = f(\alpha) + xa + r(x)$$

then it only remains to prove that $r \in R(\mathfrak{R}; E)$.

Let $\{x_n\}$ be any quasi-bounded sequence in $\mathfrak{R}$ and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n \to 0$.

$$\frac{r(\lambda_n x_n)}{\lambda_n} = \frac{1}{\lambda_n}\{f(\alpha+\lambda_n x_n) - f(\alpha) - \lambda_n x_n a\}$$

$$= x_n \left\{\frac{f(\alpha+\lambda_n x_n)-f(\alpha)}{\lambda_n x_n} - a\right\}.$$

Put

$$y_n = \frac{f(\alpha+\lambda_n x_n)-f(\alpha)}{\lambda_n x_n} - a$$

for $n=0, 1, 2, \cdots$. By assumption we have

$$\{\lim y_n\} \ni 0.$$

Since $\{x_n\}$ is a quasi-bounded sequence in $\mathfrak{R}$, by (1.9.2) it is bounded, i.e., there exists a number M such that

$$|x_n| < M, \qquad n=0, 1, 2, \cdots$$

$$\therefore \quad \left\{\lim \frac{x_n}{M} y_n\right\} \ni 0,$$

$$\therefore \quad \left\{\lim M \frac{x_n}{M} y_n\right\} \ni 0,$$

$$\therefore \quad \{\lim x_n y_n\} \ni 0.$$

Therefore

$$\left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0, \qquad \therefore \quad r \in R(\mathfrak{R}; E).$$

Thus $f : \mathfrak{R} \to E$ is differentiable at the point $\alpha \in \mathfrak{R}$, and $f'(\alpha)(x) = xa$.