149. Approximation of Transport Process by Transport Chain

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It is classical but important that the difference equations are closely related to diffusional equations. In connection with stochastic problems, this fact shows that discrete models serve as an approximation to continuous models of random motions. As is well known, the Brownian motion appears as a limit of random walk in various senses. In particular, F. Knight [3] has made pathwise approximation of Brownian motion by joining paths of random walk.

Now, let us consider the telegraph equation of infinite cable. Then it will be seen that there corresponds a stochastic process, called the transport process.

The purpose of this paper is to construct the approximate discrete chain of transport process (Theorem 1) and next, to prove, similarly to the Knight's result in Brownian motion's case, the pathwise convergence of this discrete chain to the transport process (Theorem 2).

1. Definition. Let S be the product space of one-dimensional Euclidian space E^1 and the two points set $\Theta=\{\theta=\pm 1\}$. Let $X=[X(t)=(x(t), \theta(t)), +\infty, \mathcal{M}_t, P_{(x,\theta)}, (x, \theta)\in S]$ be the right continuous strong Markov process over the state space S such that

(i) $P_{(x,\theta)}\{X(t)=(x+c\theta t, \theta)\mid t<\tau\}=1$,
where $\tau=\inf\{t: \theta(t)\neq\theta(0)\}$,

(ii) $P_{(x,\theta)}\{\tau>t\}=e^{-\kappa t}$,

(iii) $P_{(x,\theta)}\{X(\tau)=(y, -\theta')\mid X(\tau-)=(y, \theta')\}=1$.

Definition 1.1. The Markov process X is called *the transport process* (with speed c).

For simplicity, we always suppose that

Assumption 1.1. $c=1$, $\kappa=1$.

Let us denote by $\{T_t\}$ the semigroup corresponding to the transport process X, i.e.

$$T_t f(x, \theta)=E_{(x,\theta)}[f(X(t))], \tag{1.1}$$

where $E_{(x,\theta)}$ is the expectation with respect to $P_{(x,\theta)}$-measure.

Proposition 1.1. *For any given nice function f on S, $U(t, x, \theta)=T_t f(x, \theta)$ is the unique solution of the following telegraph equation:*

$$\begin{cases} \dfrac{\partial}{\partial t} U(t, x, \theta)=\theta \dfrac{\partial}{\partial x} U(t, x, \theta)-U(t, x, \theta)+U(t, x, -\theta), \\ U(t, x, \theta)\to f(x, \theta) \quad as \quad t\to 0. \end{cases} \tag{1.2}$$

Let Z be the space consisting of the countable points $(z_i, \theta) \in S$, $i = 0, \pm 1, \pm 2, \cdots, \theta = \pm 1$. Let $\boldsymbol{R} = [R(k), P^R_{(z_i,\theta)}]$ be the Markov chain on Z such that

(i) $P^R_{(z_i,\theta)}\{R(1) = (z_{i+\theta}, \theta)\} = p_{i,\theta}$,

(ii) $P^R_{(z_i,\theta)}\{R(1) = (z_i, -\theta)\} = 1 - p_{i,\theta}$.

Definition 1.2. The Markov chain $\boldsymbol{R}$ is called *the transport chain on Z.*

Proposition 1.2. *For any function f on Z, $U(k, i, \theta) = E^R_{(z_i,\theta)}[f(R(k))]$ is the unique solution of the following partial difference equation:*

$$
\left\{
\begin{aligned}
&U(k+1, z_i, \theta) - U(k, z_i, \theta) = P_{i,\theta} U(k, z_{i+\theta}, \theta) \\
&\qquad\qquad + (1 - p_{i,\theta}) U(k, z_i, -\theta) - U(k, z_i, \theta), \\
&U(0, z_i, \theta) = f(z_i, \theta).
\end{aligned}
\right.
\tag{1.3}
$$

2. Construction of approximate chain. We shall construct an approximate sequence of transport chain for the given transport process X.

Putting $Z^n = \{(i/2^n, \theta) : i = 0, \pm 1, \pm 2, \cdots, \theta = \pm 1\} (n = 1, 2, \cdots)$, we define a series of Markov times:

$$\sigma_0^n(\omega) = 0, \tag{2.1}$$

$$\sigma_1^n(\omega) = \sigma(1/2^n, \omega) \wedge \sigma^*(\omega), \tag{2.2}$$

$$\sigma_k^n(\omega) = \sigma_{k-1}^n(\omega) + \sigma_1^n(\omega^+_{\sigma^n_{k-1}}),\ k \geq 2,^{\dagger)} \tag{2.3}$$

where $\sigma(a, \omega) = \inf\{t : \theta(s, \omega) = \theta(0, \omega) \text{ for all } s \leq t$ and $|x(t, \omega) - x(0, \omega)| = a\}$,

$$\sigma^*(\omega) = \inf\{t : \theta(t, \omega) \neq \theta(0, \omega) \quad \text{and} \quad x(t, \omega) = x(0, \omega)\}.$$

Lemma 2.1. (i) $E_{(0,\theta)}[\sigma_1^n] = 1/2^n$.

$E_{(0,\theta)}[|\sigma_1^n - 1/2^n|^2] = 2^{-2n+1}$.

(ii) $E_{(0,\theta)}[|\sigma_k^n - k/2^n|^2] = \dfrac{1}{3} k 2^{-3n}$.

(iii) $P_{(0,\theta)}\{X(\sigma_1^n) = (\theta/2^n, \theta)\} = 1/(1 + 2^{-n})$.

(iv) $P_{(0,\theta)}\{\max\limits_{0 \leq k \leq K} |\sigma_k^n - k/2^n| > \varepsilon\} \leq \dfrac{1}{3} K \varepsilon^{-2} 2^{-3n}$.

Proof. Denote by $R_\lambda (\lambda > 0)$ the resolvent operator corresponding to X, i.e. $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt$. Then we have

$$
\begin{aligned}
R_\lambda f(x, \theta) = &\frac{1}{2\beta} \int_{-\infty}^\infty e^{-\beta|x-y|} \{(1+\lambda) f(y, \theta) + f(y, -\theta)\} dy \\
&- \frac{1}{2} \int_x^\infty e^{-\beta|x-y|} f(y, \theta) dy + \frac{1}{2} \int_{-\infty}^x e^{-\beta|x-y|} f(y, \theta) dy,
\end{aligned}
\tag{2.4}
$$

where $\beta = \sqrt{\lambda^2 + 2\lambda}$.

By the similar way as in [2], this formula enables us to compute the following Laplace transforms:

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†) ω^+ denotes the shifted path of ω (cf. [2]).

$$(2.5)\qquad E_{(0,\theta)}[e^{-\lambda(\sigma(a)\wedge\sigma^*)}]=\frac{(1+\lambda+\beta)e^{-\beta a}+1}{1+\lambda+\beta+e^{-\beta a}},$$

$$(2.6)\qquad E_{(0,\theta)}[e^{-\lambda(\sigma(a)\wedge\sigma^*)},\ \sigma(a)<\sigma^*]=\frac{(1+\lambda+\beta)^2-1}{(1+\lambda+\beta)^2-e^{-2\beta a}}e^{-\beta a}.$$

Thus, expanding both sides in (2.5) (respectively (2.6)) in Taylor series of λ, we can obtain the properties (i) (respectively (iii)).

Noting that σ_k^n is the sum of k-independent copies of σ_1^n with respect to $P_{(0,\theta)}$-measure, we can get the properties (ii) and, by submartingale inequality (iv). Thus we complete the proof.

Now we define the stochastic process $\boldsymbol{R}^n=[R^n(k), P_{(i/2^n,\theta)},(i/2^n,\theta)\in Z^n]$, which is desired, as follows:

$$(2.7)\qquad R^n(k)=X(\sigma_k^n)\qquad k=0,1,2,\cdots.$$

Then we have

Theorem 1. (i) *The stochastic process $\boldsymbol{R}^n$ is the transport chain which satisfies*

$$(2.8)\qquad P_{(i/2^n,\theta)}\{R^n(1)=((i+\theta)/2^n,\theta)\}=1/(1+2^{-n}),$$

$$(2.9)\qquad P_{(i/2^n,\theta)}\{R^n(1)=(i/2^n,-\theta)\}=2^{-n}/(1+2^{-n}).$$

(ii) *Let f be a function on S such that*

$$(2.10)\qquad f(x,\theta)=f(x,-\theta),$$

$$(2.11)\qquad |f(x_1,\theta)-f(x_2,\theta)|\le L|x_1-x_2|$$

for some constant L. Then it holds that

$$(2.12)\qquad \lim_{n\to\infty}E_{(j/2^m,\theta)}[f(R^n([t/2^{-n}]))]=E_{(j/2^m,\theta)}[f(X(t))]$$

uniformly in $0\le t\le T$ for each $(j/2^m,\theta)\in S$.

Proof. Property (i) is easily obtained by the strong Markov property and Lemma 2.1.

Noting that $R^n(k)=X(\sigma_k^n)$ and the transport process $\boldsymbol{X}$ is spatially homogeneous with respect to $x\in E^1$, we have for $k2^{-n}\le t<(k+1)2^{-n}$

$$(2.13)\qquad \begin{aligned}&|E_{(j/2^m,\theta)}[f(R^n([t/2^{-n}]))]-E_{(j/2^m,\theta)}[f(X(t))]|\\&=E_{(0,\theta)}[f(X(\sigma_k^n))]-E_{(0,\theta)}[f(X(\mathrm{t}))]|\\&\le LE_{(0,\theta)}[\|X(\sigma_k^n)-X(t)\|],\end{aligned}$$

where $\|\ \|$ is the quasi-metric on S obtained by

$$(2.14)\qquad \|(x_1,\theta_1)-(x_2,\theta_2)\|=|x_1-x_2|.$$

Because the speed of $\boldsymbol{X}$ is 1 (Assumption 1.1),

$$(2.15)\qquad \|X(t)-X(s)\|\le|t-s|.$$

Therefore we have by Lemma 2.1

$$(2.16)\qquad \begin{aligned}&E_{(0,\theta)}[\|X(\sigma_k^n)-X(t)\|]\\&\le E_{(0,\theta)}[|\sigma_k^n-k2^{-n}|]+|t-k2^{-n}|\\&\le(E_{(0,\theta)}[|\sigma_k^n-k2^{-n}|^2])^{1/2}+2^{-n}\\&=\left(\frac{1}{3}k2^{-3n}\right)^{1/2}+2^{-n}\\&\le 2\sqrt{T}\,2^{-n}\qquad\text{for}\quad 0\le t\le T.\end{aligned}$$

Combining (2.13) with (2.16),

(2.17) The right-hand side of (2.13) $\leq 2L\sqrt{T}\,2^{-n}$,
which implies the properties (ii).

Thus we finish the proof.

Corollary. *Let $f(x,\theta)$ be a suitably smooth function which satisfies both Conditions* (2.10) *and* (2.11). *Let $U(t, x, \theta)$ and $U^n(k, i, \theta)$ be the solutions of* (2.18) *and* (2.19) *for initial data f, respectively.*

$$(2.18)\qquad \begin{cases} \dfrac{\partial}{\partial t} U(t, x, \theta) = \theta \dfrac{\partial}{\partial x} U(t, x, \theta) - U(t, x, \theta) + U(t, x, \theta), \\ U(t, x, \theta) \to f(x, \theta) \quad \text{as} \quad t \to 0. \end{cases}$$

$$(2.19)\qquad \begin{cases} U^n(k+1, i, \theta) - U^n(k, i, \theta) = \dfrac{1}{1+2^{-n}} U^n(k, i+\theta, \theta) \\ \qquad\qquad\qquad + \dfrac{2^{-n}}{1+2^{-n}} U^n(k, i, -\theta) \\ \qquad\qquad\qquad - U^n(k, i, \theta), \\ U^n(0, i, \theta) = f^n(i, \theta), \\ \text{where} \quad f^n(i, \theta) = f(i/2^n, \theta). \end{cases}$$

Then

$$(2.20)\qquad |U^n([t/2^{-n}], [x/2^{-n}], \theta) - U(t, x, \theta)| \leq 2L\sqrt{T}\,2^{-n},$$

uniformly for $0 \leq t \leq T$ *and* $-\infty < x < \infty$.

3. Pathwise convergence. Let X be the given transport process and consider a sequence of transport chain R^n defined by (2.7). Then this sequence is approximate chains in the following sense. Define $X^n(t)$ by

$$(3.1)\qquad X^n(t) = R^n([t/2^{-n}]).$$

Then it holds that

Theorem 2. $P_{(0,\theta)}\{\lim\limits_{n\to\infty} \operatorname*{Max}\limits_{0\leq t\leq T} |X^n(t) - X(t)| = 0\} = 1$.

Proof. Let $k2^{-n} \leq t < (k+1)2^{-n}$. Then

$$(3.2)\qquad \begin{aligned} |X^n(t) - X(t)| &\leq |X(\sigma_k^n) - X(k2^{-n})| + |X(k2^{-n}) - X(t)| \\ &\leq |\sigma_k^n - k2^{-n}| + |k2^{-n} - t|, \end{aligned}$$

since $|X(t) - X(s)| \leq |t-s|$.

On the other hand, it follows from Lemma 1.1 that

$$(3.3)\qquad \begin{aligned} &P_{(0,\theta)}\{\max_{0\leq k\leq [T/2^{-n}]} |\sigma_k^n - k2^{-n}| > 2^{-n/3}\} \\ &\leq \frac{1}{3}[T/2^{-n}]2^{-7n/3} \\ &\leq \frac{1}{3}(T+1)2^{-4n/3}. \end{aligned}$$

Using Borel-Cantelli lemma, we obtain

$$(3.4)\qquad P_{(0,\theta)}\{\lim_{n\to\infty} \max_{0\leq k\leq [T/2^{-n}]} |\sigma_k^n - k2^{-n}| = 0\} = 1.$$

Combining with (2.4), we complete the proof.

References

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