146. *Characterization of a De Morgan Lattice in Terms of Implication and Negation*

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The purpose of this Note is to give a characterization of De Morgan lattice in terms of implication and negation.

The notion of De Morgan lattice has been considered by Gr. C. Moisil [4] in the work mentioned in the reference included at the end of this Note and has been studied by J. A. Kalman [3] under the name of *distributive i-lattice*. A. Bialynicki-Birula and H. Rasiowas [2] have studied this type of lattice having the first element under the name of *quasi-boolean algebra*. The nomenclature used here is due to A. Monteiro [5].

A lattice can be defined as a system $(M, \cap, \cup)$ consisting of a non empty set M and two binary operations $\cup$, $\cap$ defined on M such that the following properties are verified:

L1. $x \cup y = y \cup x$,　　L′1. $x \cap y = y \cap x$,

L2. $x \cup (y \cup z) = (x \cup y) \cup z$,　　L′2. $x \cap (y \cap z) = (x \cap y) \cap z$,

L3. $x \cup (y \cap x) = x$,　　L′3. $x \cap (y \cup x) = x$.

A lattice is called a distributive lattice if it verifies the property:

D. $x \cup (y \cap z) = (x \cup z) \cap (x \cup z)$.

A distributive lattice is called a De Morgan lattice if a unary operation $-$ is defined on it such that the following two properties hold:

M1. $--x = x$,

M2. $-(x \cup y) = -x \cap -y$.

Theorem. *Let M be a non-empty set, $\rightarrow$ a binary operation and $-$ a unary operation defined on M such that the following properties are verified:*

A1. $x \rightarrow -y = y \rightarrow -x$,

A2. $(x \rightarrow -y) \rightarrow y = y$,

A3. $(x \rightarrow y) \rightarrow z = -((-x \rightarrow z) \rightarrow -(y \rightarrow z))$.

If we write $x \cup y = -x \rightarrow y$ and $x \cap y = -(x \rightarrow -y)$, then the system $(M, \cup, \cap, -)$ is a De Morgan lattice.

Proof. M1. $x = -(-x)$.

In order to prove this, let us see the following two relations:

a) $-x \rightarrow x = x$,

b) $x \rightarrow -x = -x$.

Using twice the axiom A2 we have

$$-x\to x=((x\to -(-x))\to -x)\to x=x.$$

From a), A2, A3, A1, and A2 it follows that

$$\begin{aligned}x\to -x&=(-x\to x)\to -x=(-x\to((x\to -x)\to x))\to -x\\&=(-x\to -((-x\to x)\to -(-x\to x)))\to -x\\&=(((-x\to x)\to -(-x\to x))\to -(-x))\to -x=-x.\end{aligned}$$

We now prove 1. By successive applications of A2, A3, b), and 3) we have

$$x=(x\to -x)\to x=-((-x\to x)\to -(-x\to x))=-(-(-x\to x))=-(-x).$$

M2. $-(x\cup y)=-x\cap -y.$

$$-(x\cup y)=-(-x\to y)=-(-x\to -(-y))=-x\cap -y.$$

L1. $x\cup y=y\cup x.$

$$x\cup y=-x\to y=-x\to -(-y)=-y\to -(-x)=-y\to x=y\cup x$$

by M1 and A1.

L3. $x\cup(y\cap x)=x.$

Applying A1, M1, and A2 it follows that

$$x\cup(x\cap y)=-x\to -(x\to -y)=(x\to -y)\to -(-x)=(y\to -x)\to x=x.$$

L2. $(x\cup y)\cup z=x\cup(y\cup z).$

To prove the associative law we shall first demonstrate the following relation:

$$-x=x\to -(-(-x\to y)\to z).$$

By successive applications of A1, A3, M1, and A2 we have

$$\begin{aligned}x\to -(-(-x\to y)\to z)&=(-(-x\to y)\to z)\to -x\\&=-((-(-(-x\to y))\to -x)\to -(z\to -x))\\&=-(((-x\to y)\to -x)\to -(z\to -x))\\&=-(-x\to -(z\to -x))=-((z\to -x)\to x)=-x.\end{aligned}$$

The following formulas are proved in the same way:

$$-y=y\to -(-(-x\to y)\to z)=y\to -(-(-y\to z)\to x),$$
$$-z=z\to -(-(-y\to z)\to x).$$

Let us now prove the associative law. By definition, we have $(x\cup y)\cup z=-(-x\to y)\to z$. We substitute $-x$, y, and z for the following expressions:

$$-x=x\to -(-(-y\to z)\to x, \qquad \text{(by A1 and A2)}$$
$$y=-(y\to -(-(-y\to z)\to x)),$$
$$z=-(z\to -(-(-y\to z)\to x)).$$

Then we apply twice the axiom A3, and we have

$$\begin{aligned}(x\cup y)\cup z&=-(-x\to y)\to z=-((x\to -(-(-y\to z)\to x))\\&\quad\to -(y\to -(-(-y\to z)\to x)))\to -(z\to -(-(-y\to z)\to x))\\&=((-x\to y)\to -(-(-y\to z)\to x))\to -(z\to -(-(-y\to z)\to x))\\&=-((-(-x\to y)\to z)\to -(-(-y\to z)\to x)). \qquad (1)\end{aligned}$$

On the other hand we have $x\cup(y\cup z)=-x\to(-y\to z)$.

Substituting $-x$, $-$y and z for

$$-x=x\to-(-(-x\to y)\to z),$$
$$-y=y\to-(-(-x\to y)\to z),$$
$$z=-(z\to(-(-(-x\to y)\to z)) \qquad \text{(by A1 and A2)}$$

respectively and applying the above method we have

$$x\cup(y\cup z)=-x\to(-y\to z)$$
$$=(x\to-(-(-x\to y)\to z))\to((y\to-(-(-x\to y)\to z))$$
$$\to-(z\to-(-(-x\to y)\to z)))$$
$$=(x\to-(-(-x\to y)\to z))\to-((-y\to z)\to-(-(-x\to y)\to z))$$
$$=-((-x\to(-y\to z))\to-(-(-x\to y)\to z))$$
$$=-((-(-x\to y)\to-(-(-y\to z)\to x)). \qquad (2)$$

From (1) and (2) we have $x\cup(y\cup z)=(x\cup y)\cup z$.

D7. $x\cup(y\cap z)=(x\cup y)\cap(x\cup z)$.

Applying A1, M1, A3, and A1 it follows that

$$x\cup(y\cap z)=-x\to-(y\to-z)=(y\to-z)\to-(-x)=(y\to-z)\to x$$
$$=-((-y\to x)\to-(-z\to x))=-((-x\to y)\to-(-x\to z))$$
$$=(-x\to y)\cap(-x\to z)=(x\cup y)\cap(x\cup z).$$

The dual laws follow immediately from the above properties.

Independence of the postulates A1, A2, and A3.

1) Let A be a set consisting of two elements a and b, on which the operations $\to$ and $-$ are defined by the following tables:

$\to$	a	b
a	a	b
b	a	b

$-$	
a	b
b	a

The axioms A2 and A3 are verified but not A1 since

$$a\to-b=a \quad\text{and}\quad b\to-a=b$$

2) On the same set let us consider the operations defined by the following tables:

$\to$	a	b
a	a	a
b	a	b

$-$	
a	a
b	b

The axioms are verified but A2 since

$$(a\to-b)\to b=(a\to b)\to b=a$$

3) Finally let us consider on the same set the operations defined by the following tables:

$\to$	a	b
a	a	b
b	a	a

$-$	
a	a
b	a

A1 and A2 hold but not A3 since

$$(a\to a)\to b=a\to b=b,$$
$$-((-a\to b)\to-(a\to b)=-(b\to-b)=-a=a.$$

References

[1] G. Birkhoff: Lattice Theory. Amer. Math. Soc. (1948).
[2] A. Bialynicki-Birula and H. Rasiowa. On the representation of quasi-boolean algebras. Bull. de l'Acad. Polonaise des Sci. Classe III, **5**, 259–261 (1957).
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[5] A. Monteiro: Matrices de Morgan caracteristiques pour le calcul propositionnel classique. Anais da Ac. Brasileire de Ciencias, **32**, 1–7 (1960).