136. A New Proof of the Limit Formula of Kronecker

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Let $Q(x, y)=ax^2+2bxy+cy^2$ $(d=ac-b^2)$ be a positive definite quadratic form with discriminant $-4d$. The Epstein zeta-function of Q is defined by

$$\zeta(s, Q)=\sum_{(m,n)\neq(0,0)} Q(m, n)^{-s} \qquad (s=\sigma+it)$$

for $\sigma>1$.

It is well known that $\zeta(s, Q)$ can be continued analytically for all s-plane and has only one pole of order one at $s=1$.

The residue of this pole has been calculated by L.P. Dirichlet. And the constant term has been obtained by L. Kronecker. This is the so called Kronecker's limit formula.

These values are very important in order to calculate the class numbers of certain algebraic number fields. And so several proofs have been obtained. See for instance H. Weber [3], P. Epstein [1], and C.L. Siegel [2]. But it seems that the following proof has not been observed before.

Our method is as follows:

Theorem.

$$\zeta(s, Q)=\frac{\pi}{\sqrt{d}}\cdot\frac{1}{s-1}+2\frac{\pi}{\sqrt{d}}(\gamma+\log\sqrt{a/4d}-\log|\eta(z)|^2)+O(s-1),$$

where γ is the Euler's constant 0.57721.., and $\eta(z)$ is the Dedekind's η-function defined by

$$\eta(z)=e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz}) \qquad \text{for } \operatorname{Im}(z)>0.$$

Here $z=(-b+i\sqrt{d})/a$.

Proof. By the familliar assertion of Poisson's summation-formula we have for $\sigma>1$

$$(1) \qquad \zeta(s, Q)=2a^{-s}\zeta(2s)+2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{2\pi imx}Q(x, n)^{-s}dx,$$

where $\zeta(s)$ is the Riemann zeta-function. We denote the above integral by $I(m, n, s)$. Then by the Γ-integral, we have for $\sigma>0$

$$(2) \qquad \Gamma(s)I(m, n, s)=\int_{-\infty}^{+\infty}e^{2\pi imx}dx\int_0^{\infty}\xi^{s-1}e^{-Q(x,n)\xi}d\xi.$$

Obviously the order of integration can be changed, and so

$$(3)\qquad \Gamma(s)I(m,n,s)=\int_0^\infty \xi^{s-1}e^{-cn^2\xi}d\xi\int_{-\infty}^{+\infty}e^{2\pi imx-2bnx\xi-ax^2\xi}dx$$
$$=\sqrt{\frac{\pi}{a}}\,e^{-2\frac{b}{a}\pi mni}\int_0^\infty \xi^{s-\frac{3}{2}}e^{-\frac{d}{a}n^2\xi-\frac{\pi^2m^2}{a\xi}}d\xi.$$

Hence for $\sigma>1/2$ we have

$$(4)\qquad \Gamma(s)I(0,n,s)=\sqrt{\frac{\pi}{a}}\,\Gamma\left(s-\frac{1}{2}\right)\left(\frac{a}{d}\right)^{s-\frac{1}{2}}n^{1-2s}.$$

This gives for $\sigma>1$

$$(5)\qquad \zeta(s,Q)=2a^{-s}\zeta(2s)+2\sqrt{\frac{\pi}{a}}\left(\frac{a}{d}\right)^{s-\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1)$$
$$+2\sum_{n=1}^{\infty}\sum_{m\neq 0}I(m,n,s).$$

Now the last sum is an integral function. This can be seen from the right hand side of (3).

In order to calculate the value $I(m,n,1)$ we use Mellin's transformation-formula, that is,

$$e^{-\frac{\pi^2m^2}{a\xi}}=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(\mu)\left(\frac{\pi^2m^2}{a\xi}\right)^{-\mu}d\mu$$

for $\alpha>0$.

Inserting this into the right hand side of (3), we have

$$(6)\qquad \Gamma(s)I(m,n,s)$$
$$=\sqrt{\frac{\pi}{a}}\,e^{-2\frac{b}{a}\pi imn}\int_0^\infty \xi^{s-\frac{3}{2}}e^{-\frac{d}{a}n^2\xi}d\xi\,\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(\mu)\left(\frac{\pi^2m^2}{a\xi}\right)^{-\mu}d\mu$$
$$=\sqrt{\frac{\pi}{a}}\,e^{-2\frac{b}{a}\pi imn}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(\mu)\left(\frac{\pi^2m^2}{a}\right)^{-\mu}d\mu\int_0^\infty \xi^{s+\mu-\frac{3}{2}}e^{-\frac{d}{a}n^2\xi}d\xi.$$

The change of the order of integration can be justified under the condition $\sigma+\alpha>1/2$.

Hence we have

$$(7)\qquad \Gamma(s)I(m,n,s)=\sqrt{\frac{\pi}{a}}\,e^{-2\frac{b}{a}mn\pi i}\frac{1}{2\pi i}$$
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(\mu)\Gamma\left(s+\mu-\frac{1}{2}\right)\left(\frac{\pi^2m^2}{a}\right)^{-\mu}\left(\frac{dn^2}{a}\right)^{-(s+\mu-\frac{1}{2})}d\mu.$$

Now we remember the formula

$$\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)=2\sqrt{\pi}\,2^{-2\mu}\Gamma(2\mu),$$

and so we have

$$(8)\qquad I(m,n,1)=2\frac{\pi}{\sqrt{d}}e^{-2\frac{b}{a}mn\pi i}\cdot\frac{1}{n}\cdot\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(2\mu)(4\pi^2(mm)^2d/a^2)^{-\mu}d\mu.$$

Again by virtue of the Mellin's transformation-formula we have

$$(9)\qquad I(m,n,1)=\frac{\pi}{n\sqrt{d}}e^{-2\frac{b}{a}mn\pi i-2\frac{\sqrt{d}}{a}\pi n|m|}$$

$$=\begin{cases}\dfrac{\pi}{n\sqrt{d}}e^{2\pi imnz} & (m>0)\\[2ex] \dfrac{\pi}{n\sqrt{d}}e^{2\pi imn\bar{z}} & (m<0).\end{cases}\qquad (z=(-b+i\sqrt{d})/a)$$

This gives

$$(10)\qquad \sum_{n=1}^{\infty}\sum_{m\neq 0}I(m,n,1)=-\frac{\pi}{\sqrt{d}}\log\prod_{m=1}^{\infty}(1-e^{2\pi imz})(1-e^{-2\pi im\bar{z}})$$

$$=-\frac{\pi^2}{6a}-\frac{\pi}{\sqrt{d}}\log|\eta(z)|^2.$$

Now

$$\Gamma\left(s-\frac{1}{2}\right)=\sqrt{\pi}-\sqrt{\pi}(\gamma+\log 4)(s-1)+O((s-1)^2),$$

$$\Gamma(s)=1-\gamma(s-1)+O((s-1)^2),$$

and

$$\zeta(2s-1)=\frac{1}{2(s-1)}+\gamma+O((s-1))\qquad (s\to 1).$$

(See Chap. XII and XIII of [4].)

Collecting these formulas we have

$$(11)\qquad 2\sqrt{\pi/a}\,\Gamma\left(s-\frac{1}{2}\right)\Gamma^{-1}(s)(a/d)^{s-\frac{1}{2}}\zeta(2s-1)$$

$$=\frac{\pi}{\sqrt{d}}\cdot\frac{1}{s-1}+\frac{2\pi}{\sqrt{d}}\left(\gamma+\log\sqrt{\frac{a}{4d}}\right)+O((s-1))\qquad (s\to 1).$$

Hence from (5), (10), and (11) the theorem follows.

References

[1] P. Epstein: Zur theorie allgemeiner Zeta-funktion. I. Math. Ann., **56**, 615–644 (1903).

[2] C. L. Siegel: Lectures on advanced analytic number theory. Tata Inst., Bombay (1961).

[3] H. Weber: Lehrbuch der algebra. III. Braunschweig (1908).

[4] E. T. Whittaker and G. N. Watson: A Course of Modern Analysis. Cambridge (1927).