135. Notes on the Uniform Distribution of Sequences of Integers

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In 1961 I. Niven [5] introduced the following concept of uniform distribution of sequences of integers. Let $A=(a_n)$ be an infinite sequence of integers not necessarily distinct from each other. For any integers j and $m\geqq 2$ we denote by $A(N, j, m)$ the number of terms a_n $(1\leqq n\leqq N)$ satisfying the condition $a_n\equiv j \pmod{m}$. The sequence A is said to be uniformly distributed (mod m) if the limit

$$\lim_{N\to\infty}\frac{1}{N}A(N, j, m)=\frac{1}{m}$$

exists for all j, $1\leqq j\leqq m$. If A is uniformly distributed (mod m) for every integer $m\geqq 2$, A is said to be uniformly distributed.

S. Uchiyama [9] has proved the following theorem which is the analogue of the Weyl criterion:

Theorem 1. *Let $A=(a_n)$ be an infinite sequence of integers. A necessary and sufficient condition that A be uniformly distributed* (mod m), $m\geqq 2$, *is that*

$$\lim_{N\to\infty}\frac{1}{N}S_N\left(A,\frac{h}{m}\right)=0$$

for all h, $1\leqq h\leqq m-1$, where

$$S_N(A, t)=\sum_{n=1}^{N}e(a_n t),\qquad e(t)=e^{2\pi it}.$$

Hence:

Corollary 1. *A necessary and sufficient condition for an infinite sequence $A=(a_n)$ of integers to be uniformly distributed is that*

$$\lim_{N\to\infty}\frac{1}{N}S_N(A, t)=0$$

for all rational numbers t, $t\not\equiv 0$ (mod 1).

In order to prove Theorem 1 it will suffice to observe that

$$\sum_{h=1}^{m-1}\left|S_N\left(A,\frac{h}{m}\right)\right|^2=m\sum_{j=1}^{m}\left(A(N, j, m)-\frac{N}{m}\right)^2.$$

The notion of uniform distribution of integers has been generalized by H. G. Meijer [4] to the notion of uniform distribution of g-adic

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numbers and by L. A. Rubel [7] to the notion of uniform distribution of elements in locally compact topological groups.

1. The following theorem, which is due to C. L. van den Eynden [2], expresses a connection between the uniform distribution of sequences of integers and the classical uniform distribution (mod 1) of sequences of real numbers.

Theorem 2. *If* (α_n) *is an infinite sequence of real numbers such that the sequence* (α_n/m) *is uniformly distributed* (mod 1) *for all integers* $m\neq 0$, *then the sequence of the integer parts* $([\alpha_n])$ *is uniformly distributed.*

This result has a number of applications.

Corollary 2. *Let* (α_n) *be a sequence of real numbers. If the sequence* $(\alpha_n s)$ *is uniformly distributed* (mod 1) *for every real number* $s\neq 0$, *then the sequence of integers* $([\alpha_n s])$ *is uniformly distributed for every real number* $s\neq 0$.

Proof. From the assumption on (α_n) it follows that the sequence $(\alpha_n s/m)$, where m is any integer $\neq 0$, is uniformly distributed (mod 1) for every real $s\neq 0$.

By a similar reasoning we obtain

Corollary 3. *Let* (α_n) *be a sequence of real numbers. If the sequence* $(\alpha_n s)$ *is uniformly distributed* (mod 1) *for almost all real numbers* s, *then the sequence of integers* $([\alpha_n s])$ *is uniformly distributed for almost all real numbers* s.

Now, if $f(t)$ is a real function defined for $t>0$, then the behaviour of the residues of the numbers $f(n)$ (mod 1) for positive integral n with respect to their distribution on the unit-interval can in many cases be derived from the properties of the derivative $f'(t)$ (if $f(t)$ is a differentiable function) or from the properties of the difference $\Delta f(t) = f(t+1)-f(t)$. It occurs often that these properties are not violated if one considers constant multiples of $f(t)$. Hence applying Theorem 2, or Corollary 2, we have the following result.

Theorem 3. (a) *If* $f(t)$ $(t>0)$ *is differentiable, and if* $f(t)\to\infty$, $f'(t)\to 0$ *(monotonically) and* $tf'(t)\to\infty$ *as* $t\to\infty$, *then the sequence of integers* $([f(n)])$ *is uniformly distributed.*

(b) *If the sequence* $(f(n))$ *has the property that* $\Delta f(n)=f(n+1)-f(n)\to\theta$ *(irrational) as* $n\to\infty$, *then the sequence* $([f(n)])$ *is uniformly distributed.*

(c) *If the sequence* $(f(n))$ *has the property that* $\Delta f(n)\to 0$ *(monotonically) and* $n|\Delta f(n)|\to\infty$ *as* $n\to\infty$, *then the sequence* $([f(n)])$ *is uniformly distributed.*

It is well known that the sequence $(n^\sigma s)$ is uniformly distributed (mod 1) for every real $s\neq 0$ and σ with $0<\sigma<1$, as has been shown by several authors (see for this result and its generalizations [1, p. 9]). Thus, by Corollary 2, the sequence of integers ($[n^{\sigma}s]$) is uniformly distributed for every real $s\neq 0$ and σ, $0<\sigma<1$. This result, the special case of which for $\sigma=1/q$, q an integer $\geqq 2$, is due to S. Uchiyama [9], can also be derived by applying Theorem 3.

The following theorem is also due to van den Eynden [2].

Theorem 4. *A necessary and sufficient condition for a real sequence* (α_n) *to be uniformly distributed* (mod 1) *is that the sequence of integer parts* ($[m\alpha_n]$) *is uniformly distributed* (mod m) *for all integers* $m\geqq 2$.

Proof. Let j be any one of the residues $0, 1, \cdots, m-1$ (mod m). Then the set of positive integers

$$\{n:[m\alpha_n]\equiv j \pmod m\}=\left\{n:\frac{j}{m}\leqq \alpha_n-[\alpha_n]<\frac{j+1}{m}\right\}.$$

Note that any interval $[\alpha, \beta)\subset[0, 1)$ can be approximated by intervals of the type $[j/m, (j+1)/m)$ ($m=2, 3, \cdots; j=0, 1, \cdots, m-1$) as closely as we want.

2. It is well known that the sequence ($\log n$) is not uniformly distributed (mod 1). In order to show that the sequence ($[\log n]$) is not uniformly distributed (mod m) for any $m\geqq 2$ we apply Corollary 1.

Theorem 5. *The sequence of integers* $A=([\log n])$ *is not uniformly distributed* (mod m) *for any* $m\geqq 2$.

Proof. Set $b=[\log N]$. Let k be any integer between 0 and b, and $B(k)$ be the number of solutions in positive integers n of the equation $[\log n]=k$. Then we have $B(k)=e^{k+1}-e^k+O(1)$, so that (for t rational and $\not\equiv 0$ (mod 1))

$$\frac{1}{N}S_N(A, t)=\frac{1}{N}\sum_{k=0}^{b}B(k)e(kt)$$
$$=\frac{e-1}{N}\sum_{k=0}^{b}e^{k(1+2\pi it)}+\frac{1}{N}\sum_{k=0}^{b}O(1).$$

The second term on the right goes to 0 as $N\to\infty$, but the first term does not as one can easily show (in fact, this term is bounded but does not converge as $N\to\infty$).

3. As has been pointed out by G. Helmberg [3] Theorem 3 (or its proof given, at least) in [9, §4] was wrong. In order to state a correct version of this theorem we recall the concept of 'almost uniform distribution (mod 1)' of sequences of real numbers, which has been introduced by I. I. Pjateckiĭ-Šapiro [6]. An infinite sequence (α_n) of real numbers is said to be almost uniformly distributed (mod 1), if there is a strictly increasing sequence (N_k) of natural numbers such that if $P_k(\alpha, \beta)$ denotes the number of terms α_n ($1\leqq n\leqq N_k$) satisfying the condition $\alpha\leqq\alpha_n<\beta$ (mod 1), where $0\leqq\alpha<\beta\leqq 1$, then for any such fixed real numbers α, β we have

$$\lim_{k\to\infty}\frac{1}{N_k}P_k(\alpha,\beta)=\beta-\alpha.$$

Theorem 6. *If $A=(a_n)$ is a uniformly distributed sequence of integers, then the sequence $(a_n s)$ is almost uniformly distributed* (mod 1) *for almost all real numbers s.*

Proof. As in [9, §4] we have for any integer $h\neq 0$

$$\int_0^1\left|\frac{1}{N}S_N(A,ht)\right|^2 dt\leqq\frac{2}{m}+\frac{2}{N^2}D^2(N,m),$$

where $m\geqq 2$ and

$$D^2(N,m)=\sum_{j=1}^{m}\left(A(N,j,m)-\frac{N}{m}\right)^2.$$

Since A is uniformly distributed (mod m) for every $m\geqq 2$, we find

$$\limsup_{N\to\infty}\int_0^1\left|\frac{1}{N}S_N(A,ht)\right|^2 dt\leqq\frac{2}{m},$$

and so

$$\lim_{N\to\infty}\int_0^1\left|\frac{1}{N}S_N(A,ht)\right|^2 dt=0,$$

whence

$$\liminf_{N\to\infty}\left|\frac{1}{N}S_N(A,hs)\right|=0$$

for almost all real s. The result follows from this at once.

Our proof of Theorem 6 can easily be extended to prove

Theorem 7. *Let $f(t)\in L^2$ be a function with period* 1 *and mean value* 0 (*i.e.*, $\int_0^1 f(t)dt=0$). *Then for any uniformly distributed sequence (a_n) of integers we have*

$$\lim_{N\to\infty}\int_0^1\left|\frac{1}{N}\sum_{n=1}^{N}f(a_n t)\right|^2 dt=0.$$

We now introduce the notion of 'almost uniform distribution' of sequences of integers. An infinite sequence $A=(a_n)$ of integers is said to be almost uniformly distributed (mod m), where $m\geqq 2$, if there is a strictly increasing sequence (N_k) of natural numbers such that

$$\lim_{k\to\infty}\frac{1}{N_k}A(N_k,j,m)=\frac{1}{m}$$

for all j, $1\leqq j\leqq m$. If A is almost uniformly distributed (mod m) for every $m\geqq 2$, we say that A is almost uniformly distributed.

A necessary and sufficient condition for a sequence $A=(a_n)$ of integers to be almost uniformly distributed (mod m), $m\geqq 2$, is that

$$\liminf_{N\to\infty}\frac{1}{N^2}\sum_{h=1}^{m-1}\left|S_N\left(A,\frac{h}{m}\right)\right|^2=0.$$

As an application of Theorem 6 we shall mention the following

Theorem 8. *If* (a_n) *is a uniformly distributed sequence of integers, then the sequence of integers* $([a_n s])$ *is almost uniformly distributed for almost all real members s.*

Proof. By Theorem 6, the sequence $(a_n s)$ is almost uniformly distributed (mod 1) for almost all real s. The rest of the proof can easily be carried out as in Proof of Corollary 3 (or Corollary 2).

4. In connection with Theorem 6 the following theorem will be of some interest.

Theorem 9. *Let* $A=(a_n)$ *be a sequence of integers not necessarily distinct from each other. If the condition*

$$\sum_{2\leq m\leq M} mD^2(N, m)=O(MN^2)$$

is fulfilled for all large N and some $M\geqq(\log N)^{1+\varepsilon}$ $(\varepsilon>0)$, *where*

$$D^2(N, m)=\sum_{j=1}^{m}\left(A(N, j, m)-\frac{N}{m}\right)^2,$$

then the sequence $(a_n s)$ *is uniformly distributed* (mod 1) *for almost all real numbers s.*

Proof. We infer from the proof of Theorem 6 that for any integer $h\neq 0$

$$\int_0^1\left|\frac{1}{N}S_N(A, ht)\right|^2 dt=O\left(\frac{1}{M}\right)=O\left(\frac{1}{(\log N)^{1+\varepsilon}}\right).$$

One may proceed the proof henceforward in a standard way.

5. A result in a somewhat different direction is the following

Theorem 10. *Let* $A=(a_n)$ *be a uniformly distributed sequence of non-negative integers all different from each other, and let A be measurable in the sense of* [8, §2] (*Banach-Buck measure*). *Then for all but possibly countably many real numbers s the sequence* $(a_n s)$ *is almost uniformly distributed* (mod 1).

Proof. We use part of Corollary 1 of [8], namely: if $A=(a_n)$ is uniformly distributed and if A is Banach-Buck measurable, then this measure of A equals 1. It is then obvious that the sequence A has positive density. Our theorem is thus an easy consequence of a result of Pjateckiĭ-Šapiro [6].

References

[1] J. Cigler and G. Helmberg: Neuere Entwicklungen der Theorie der Gleichverteilung. Jahresber. Deutsch. Math. Verein., **64**, Heft, 1, 1–50 (1962).

[2] C. L. van den Eynden: The Uniform Distribution of Sequences. Dissertation, Univ. of Oregon (1962).

[3] G. Helmberg: Review for the paper [9] below. Zbl. f. Math., **131**, 292 (1966).

[4] H. G. Meijer: Uniform Distribution of g-adic Numbers. Dissertation, Univ. of Amsterdam (1967).

[5] I. Niven: Uniform distribution of sequences of integers. Trans. Amer. Math. Soc., **98**, 52–61 (1961).
[6] I. I. Pjateckii-Šapiro: On a generalization of the notion of uniform distribution of fractional parts. Matem. Sbornik, **30(72)**, 669–676 (1952) (in Russian).
[7] L. A. Rubel: Uniform distribution in locally compact groups. Comment. Math. Helv., **39**, 253–258 (1965).
[8] M. Uchiyama and S. Uchiyama: A characterization of uniformly distributed sequences of integers. J. Fac. Sci., Hokkaidô Univ., Ser. I, **16**, 238–248 (1962).
[9] S. Uchiyama: On the uniform distribution of sequences of integers. Proc. Japan Acad., **37**, 605–609 (1961).