130. A Note on Semi-prime Modules. I

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Feller and Swokowski [1, 2] have generalized Goldie's works on prime and semi-prime rings [3, 4] to modules.

It is the aim of the present note to investigate these modules and semi-prime Goldie rings. This note lays a result concerning the dimensions of semi-prime modules and of semi-prime Goldie rings. We can prove that the set of the subisomorphism classes of basic submodules of a semi-prime R-module M corresponds one-to-one onto the set of the minimal annihilator ideals of the semi-prime Goldie ring R (see Theorem 7) under no maximum conditions for right complements and for right annihilators of M. The relationship between prime and semi-prime modules is also studied, and Theorem 8 shows that clM_i is a prime R_i-module, where clM_i is a homogeneous component of M, and R_i is the minimal annihilator ideal of R which corresponds to clM_i.

Throughout this paper, R will denote a right Goldie ring; that is

(a) R satisfies the maximum condition for right complements;

(b) R satisfies the maximum condition for right annihilators.

All R-modules will mean faithful right R-modules. If M and N are R-modules, then M is an essential extension of N if $N \subseteq M$ and $N \cap L \neq 0$ for every non-zero submodule L of M. In this case, we call N a large submodule of M. We shall also speak of large right ideals of R by considering R as a right module over itself. Let M be an R-module and let X and Y be subsets of M and R respectively, then the annihilators are defined as $X_r = \{a \in R \mid xa = 0 \text{ for all } x \in X\}$ and $Y_l = \{m \in M \mid my = 0 \text{ for all } y \in Y\}$. The closure clN of a submodule N of M is defined by $clN = \{m \in M \mid mL \subseteq N : L \text{ a large right ideals of } R\}$. If $clN = N$, then N is said to be closed. If R is a semi-prime Goldie ring, then according to Theorem 5 in [4], a right ideal of R is large if and only if it contains a regular element. Hence, in this case, $clN = \{m \in M \mid mc \in N : c \text{ a regular element of } R\}$. The singular submodule $M^\blacktriangle$ of M is defined as $cl0$. Let A be a right ideal of R. Then the singular submodule of A-module M is denoted by $(M_A)^\blacktriangle$. As in [2], an R-module M is said to be semi-prime if the prime radical $P(M)$[1)]

1) Cf. [2, p. 825].

of M is zero. It follows from the definition that $M^{\blacktriangle}\subseteq P(M)$ and consequently if M is semi-prime, then we have $M^{\blacktriangle}=0$. "submodule" and "homomorphism" will mean always "R-submodule" and "R-homomorphism" respectively.

From Proposition 3.5 and Theorem 4.8 in [2] we have

Proposition 1. *Let R be a Goldie ring and let M be a faithful R-module with $M^{\blacktriangle}=0$. Then M is semi-prime if and only if R is semi-prime.*

As in [5, p. 270] we say that R-modules M and N are subisomorphic if there exist isomorphisms θ and ϕ such that $\theta M\subseteq N$ and $\phi N\subseteq M$. A submodule B of an R-module M is said to be basic if B is subisomorphic to each of its submodules and $B^{\blacktriangle}=0$. A right ideal J of a ring R is said to be basic if J is a basic submodule when it is considered as a right R-module.

Proposition 2. *Let M be a semi-prime R-module. Then*

(i) *Every non-zero submodule of M contains a basic submodule.*

(ii) *Every basic submodule of M is subisomorphic to a basic right ideal of R and conversely.*

Proof. (i): Let J be a direct sum of uniform right ideals J_i $(i=1,\cdots,n)$ and let J be large in R. If N is a non-zero submodule of M and if n is a non-zero element of N, then we have $nJ\neq 0$, because $M^{\blacktriangle}=0$. Hence $nJ_i\neq 0$ for some i. By Theorem 2.4 in [1], we have $J_i\cong nJ_i$. However, by Lemma 3.1 in [5], J_i is a basic right ideal and thus nJ_i is a basic submodule contained in N. (ii): Since M is faithful, applying the method of proof of part (i), it will be evident.

Clearly a submodule of a basic module is also basic. If M is a semi-prime module, then a basic submodule of M is uniform. For, by Proposition 2 a basic submodule contains a uniform submodule and is isomorphic to a submodule of the latter, so is itself uniform.

From the definition of basic submodules we have easily

Lemma 3. *Let M be a semi-prime uniform R-module and let B and B' be basic submodules of M. Then B and B' are subisomorphic.*

We now classify the uniform submodules of M as follows: Uniform submodules U and V are said to be related (in symbol: $U\sim V$), provided that basic submodules contained in U and V are subisomorphic. In particular, if U and V are basic submodules, then $U\sim V$ if and only if U and V are subisomorphic.

Proposition 4.[2] *Let M be a finite dimensional semi-prime R-module in the sense of Goldie* [4] *and $n=\dim M$.*[3] *Suppose that both $U_1+\cdots+U_n$ and $V_1+\cdots+V_n$ are direct sums of uniform submodules*

2) Cf. [7].
3) Cf. [4, p. 202].

of M, then it is possible to number the direct summands in such a way that $U_i \sim V_i$ ($i=1, \cdots, n$).

Proof. Let $S_1 = U_2 \oplus \cdots \oplus U_n$. If $S_1 \cap V_i \neq 0$ for all i ($i=1, \cdots, n$), then, by Theorem 1.1 in [4], S_1 is large in M. Hence we have $n-1 = \dim M$, which is a contradiction. Thus there exists V_i such that $S_1 \cap V_i = 0$. We renumber so that $S_1 \cap V_1 = 0$. Since $S_1 \oplus V_1$ is large, we have $(S_1 \oplus V_1) \cap U_1 \neq 0$. By Proposition 2 $(S_1 + V_1) \cap U_1$ contains a basic submodule B. For each $b \in B$, we write $b = v_1 + u_2 + \cdots + u_n$, where $v_1 \in V_1$ and $u_i \in U_i$. As b runs over B, the map $\theta : b \to v_1$ is a homomorphism of B into V_1. By Lemma 5.4 in [5], θ is either zero or an isomorphism. If $\theta = 0$, then we have $B \subseteq U_2 + \cdots + U_n$, a contradiction. Hence θ is an isomorphism and thus we have $U_1 \sim V_1$. Since $V_1 \oplus U_2 \oplus \cdots \oplus U_n$ is large, we can repeat the process with this direct sum instead of $U_1 \oplus \cdots \oplus U_n$. Set $S_2 = V_1 \oplus U_3 \oplus \cdots \oplus U_n$ and replace U_2 by V_2 (renumbering U_i if necessary). Then $V_1 \oplus V_2 \oplus U_3 \oplus \cdots \oplus U_n$ is large and $U_2 \sim V_2$. Continuing in this way, we obtain $U_1 \sim V_1, \cdots \ U_n \sim V_n$.

Corollary. *Let R be a semi-prime Goldie ring and $n = \dim R$. Suppose that both $I_1 + \cdots + I_n$ and $J_1 + \cdots + J_n$ are direct sums of uniform right ideals of R, then it is possible to number the direct summands in such a way that $I_i \sim J_i$ ($i=1, \cdots, n$).*

If N is a submodule of a semi-prime R-module M and Q is the right quotient ring of R, then as in [6, p. 134] N can be imbedded in the Q-module $N \otimes_R Q \cong NQ$. The elements of NQ may be written in the form nc^{-1} for an element $n \in N$ and for a regular element $c \in R$, and we may assume that $N \subseteq NQ$, where $n \in N$ is identified with $n \cdot 1 \in NQ$.

Let R be a semi-prime Goldie ring with the right quotient ring Q and let $R_1, \cdots, R_t$ be the minimal annihilator ideals of R. Then, according to [4], $Q_1 = R_1 Q, \cdots, Q_t = R_t Q$ are the minimal ideals of Q and R_i is a Goldie prime ring with the right quotient ring Q_i ($i=1, \cdots, t$).

Proposition 5. *Let R be a semi-prime Goldie ring and let I, J be uniform right ideals of R. Then the following conditions are equivalent.*

(i) $IQ \cong JQ$,

(ii) $I \sim J$,

(iii) $I_r = J_r$,

(iv) *$I, J \subseteq R_i$ for some i ($1 \leqq i \leqq t$).*

Proof. (i)$\Rightarrow$(ii): Let θ be an isomorphism of IQ onto JQ. Then we can show that I and J contain isomorphic non-zero right ideals $\theta^{-1}(\theta I \cap J)$ and $I \cap J$ respectively. Hence $I \sim J$. (ii)$\Rightarrow$(iii) will be seen by the property of subisomorphic. (iii)$\Rightarrow$(iv): By Theorem 5.1 in [4], there exist i, k, such that $I \subseteq R_i$ and $J \subseteq R_k$. Since R is semi-prime, it follows that $I_r \supsetneqq R_1 \oplus \cdots \oplus R_{i-1} \oplus R_{i+1} \oplus \cdots \oplus R_t$, $I_r \not\supseteq R_i$ and that $J_r \supsetneqq R_1 \oplus \cdots \oplus R_{k-1} \oplus R_{k+1} \oplus \cdots \oplus R_t$, $J_r \not\supseteq R_k$. Hence we have $i=k$. (iv)$\Rightarrow$(i): It follows from the fact that IQ is a minimal right ideal and R_iQ is a simple component of Q.

Consider the set of basic submodules of M and the equivalence classes of basic submodules under subisomorphism. We denote by $\{B\}$ the class to which the basic submodule B belongs and by M_B the sum of all $B' \in \{B\}$. Then we have $(M_B)_r = B'_r$ for every $B' \in \{B\}$.

Lemma 6. *Let B' be a basic submodule contained in M_B. Then $B' \in \{B\}$.*

Proof. Suppose that $B' \notin \{B\}$, then by Propositions 2 and 5 there exist uniform right ideals J, J' in R_i, R_j $(i \neq j)$ respectively such that $J \sim B$ and $J' \sim B'$. Hence $B'_r = J'_r \not\supseteq R_j$ and $J_r = B_r \supsetneqq R_j$. However, by the assumption, $B'_r \supseteq (M_B)_r = B_r \supsetneqq R_j$. This is a contradiction.

From Propositions 2 and 5 M has only a finite number k of subisomorphism classes of basic submodules and k is equal to t, where t is the number of the minimal annihilator ideals of R. Let $M_1, \cdots, M_t$ be the corresponding sum of submodules of these classes. Then, by Propositions 2 and 5 there is one-to-one correspondence, in the sense of subisomorphism, between $\{M_i\}$ and $\{R_i\}$.

In the remainder of this paper, R_i will denote a minimal annihilator ideal of R which corresponds to M_i $(i=1, \cdots, t)$.

Then we have the following properties: $(B_i)_r = (M_i)_r = (J_i)_r \supsetneqq R_j$ $(i \neq j)$ and $\not\supseteq R_i$, where B_i is a basic submodule in M_i and J_i is a uniform right ideal in R_i $(i=1, \cdots, t)$.

Now suppose that $M_1 \cap (M_2 + \cdots + M_t) \neq 0$, then there exists a basic submodule B contained in $M_1 \cap (M_2 + \cdots + M_t)$. By Lemma 6 and by the above note, $B_r \not\supseteq R_1$. However, by the above note we have $B_r \supsetneqq R_1$, which is a contradiction. Thus $M_1 + \cdots + M_t$ is a direct sum. Moreover $M_1 \oplus \cdots \oplus M_t$ is large, because otherwise there would be a basic submodule B such that $B \cap (M_1 \oplus \cdots \oplus M_t) = 0$ which cannot hold. We have therefore

Theorem 7. *Let M be a semi-prime R-module. Then*

(i) *M has only a finite number k of subisomorphism classes of basic submodules and k is equal to t, where t is the number of the minimal annihilator ideals of R.*

(ii) *If $M_1, \cdots, M_t$ are the corresponding sum of submodules of these classes, then $M_1 + \cdots + M_t$ is a direct sum, which is a large submodule of M.*

As in [1], an R-module M is said to be prime if $N_r = 0$ for every non-zero submodule N of M and $M^{\blacktriangle} = 0$.

Corollary. *Let M be a semi-prime R-module. Then M is prime* *if and only if* $t=1$.

Proof. Suppose that M is prime, then R is a prime Goldie ring by Proposition 1.2 in [1]. Hence, by Lemma 3.1 in [5] and Theorem 7, we have $t=1$. The "only if" part follows from Lemma 3.1 in [5], Theorem 7 and Proposition 1.3 in [1].

From Proposition 4.1 in [2] $clM_1+\cdots+clM_t$ is also a direct sum. And it will be proved, by the next theorem, that $U\sim B_i$ if and only if $U\subseteq clM_i$, where U is a uniform submodule and where B_i is a basic submodule contained in M_i. clM_i is a homogeneous component of M $(i=1,\cdots,t)$.

Theorem 8. *Let M be a semi-prime R-module. Then*

(i) *For each uniform submodule U, $U\subseteq clM_i$ if and only if $U\sim B_i$, where B_i is a basic submodule contained in M_i.*

(ii) *M_i is a prime R_i-module $(i=1,\cdots,t)$.*

(iii) *clM_i is a prime R_i-module $(i=1,\cdots,t)$.*

Proof. (i): Suppose that $U\sim B_i$ then, by the definition, there exists a basic submodule B in U such that $B\sim B_i$. Hence $B\in M_i$. By proposition 4.1 in [2], we have $U\subseteq clU=clB\subseteq clM_i$. Conversely suppose that $U\subseteq clM_i$ and that $U\sim B_k$ $(i\neq k)$, where B_k is a basic submodule in M_k, then, we have $U\subseteq clM_k$, a contradiction. Hence $U\sim B_i$. (ii): We prove that M_1 is a prime R_1-module. In view of Proposition 1.3 in [1], it is enough to show that the singular submodule $(M_{1R_1})^{\blacktriangle}=0$. Let m be an element in $(M_{1R_1})^{\blacktriangle}$. Then there exists a regular element c_1 of R_1 such that $mc_1=0$. Now put $c=c_1+c_2+\cdots+c_t$, where c_i $(i=2,\cdots,t)$ is any regular element of R_i. Then c is a regular element of R and we have $mc=0$ since $m_r\supseteq R_i$ $(i=2,\cdots,t)$. Thus $m\in M^{\blacktriangle}$ and therefore we have $(M_{1R_1})^{\blacktriangle}=0$, as desired. (iii): We prove that clM_1 is a prime R_1-module. By Proposition 4.1 in [2], we have $M_1Q_1=M_1Q=cl(M_1)Q\supseteq clM_1$. Let $x=m_1c_1^{-1}$ be an element in clM_1, where $m_1\in M_1$ and c_1 is a regular element of R_1. Then $xc_1=m_1$. Now suppose that $x\in(clM_1)^{\blacktriangle}_{R_1}$, then there exists a regular element d_1 of R_1 such that $xd_1=0$. For elements c_1, d_1, there exist regular elements c_1', d_1' of R_1 such that $c_1c_1'=d_1d_1'$. Then we have $m_1c_1'=xc_1c_1'=xd_1d_1'=0$ and thus $m_1\in(M_{1R_1})^{\blacktriangle}=0$. Hence we have $x=0$, which completes the proof.

Remark. Let R be a semi-prime Goldie ring and consider R as a right R-module. Then $clM_i=R_i$ $(i=1,\cdots,t)$. For, by Proposition 5, $M_i\subseteq R_i$. Hence $clM_i\subseteq clR_i=R_i$, since R_i is a complement right ideal in the sence of Goldie [4]. Conversely let $U_1\oplus\cdots\oplus U_l$ be a large right ideal in R_i, where U_i is a uniform right ideal. Then, by Lemma 3.10 in [4], $cl(U_1\oplus\cdots\oplus U_l)=R_i$. On the other hand, by Theorem 8, $U_i\subseteq clM_i$ and thus $cl(U_1\oplus\cdots\oplus U_l)\subseteq clM_i$. Hence we have $clM_i\subseteq R_i$.

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