182. *On the Type of an Associative H-space of Rank Three*

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1. Introduction. A topological space X with a continuous multiplication with unit is called an H-space. If this multiplication is associative, X is called an associative H-space. Suppose that X is an associative H-space and that the integral cohomology of X is finitely generated. Then it follows from the classical Hopf's theorem that the rational cohomology of X is an exterior algebra on a finite number of odd dimensional generators. The number of such generators is called the rank of X. The dimensions in which the generators occur is called the type of X. L. Smith determined all the possible types of associative H-spaces of rank 2 [6].

In this paper, we apply L. Smith's method to an associative H-space of rank 3 and determine the types of such a space.

Theorem. *Let X be an arcwise connected H-space of rank* 3 *with $H_*(X; Z)$ finitely generated as an abelian group. Then the type of X is either* (1, 1, 1), (1, 1, 3), (1, 3, 3), (1, 3, 5), (1, 3, 7), (1, 3, 11), (3, 3, 3), (3, 3, 5), (3, 3, 7), (3, 3, 11), (3, 5, 7), (3, 7, 11), (3, 5, 5), (3, 5, 11), (3, 7, 7), *or* (3, 11, 11).

Examples of H-spaces having the types from (1, 1, 1) to (3, 7, 11) are given by $S^1\times S^1\times S^1$, $S^1\times S^1\times S^3$, $S^1\times S^3\times S^3$, $S^1\times SU(3)$, $S^1\times SP(2)$, $S^1\times G_2$, $S^3\times S^3\times S^3$, $S^3\times SU(3)$, $S^3\times SP(2)$, $S^3\times G_2$, $SU(4)$, and $SP(3)$ respectively.

The author does not know whether the remaining types are realized or not.[1] I wish to express my hearty thanks to L. Smith for suggesting this problem and giving me many helpful advices, and to Professors K. Morita and R. Nakagawa for their criticism and encouragement.

2. Some results on unstable polyalgebras. A polynomial algebra R over the mod p Steenrod algebra A_p(p: prime) is called an unstable polyalgebra over A_p, if it is an algebra that is left A_p-module satisfying

1) After the manuscript had been submitted, L. Smith suggested to me that it was possible to show that an H-space of type (3, 5, 5), (3, 5, 11), (3, 7, 7) or (3, 11, 11) did not exist. In fact, the non-existence of H-spaces with the types (3, 5, 11) and (3, 11, 11) is proved by using the Steenrod operation P^2.

(1) $P_p^n x = 0$ if $2n > \deg x$

(2) $P_p^n x = x^p$ if $2n = \deg x$,

where we denote Sq^{2m} by P^m if $p=2$. This terminology is due to L. Smith [6].

Next Theorem is the result of A. Clark [1].

Theorem 2.1. *Let R be an unstable polyalgebra over A_p (p : prime). If $2m$, $m \not\equiv 0 \bmod p$ is the degree of a generator of R, then R has a generator in some degree $2n$ for which $n \equiv 1-p \bmod m$.*

Corollary 2.2. *Let R be an unstable polyalgebra over A_p (p : odd prime) on three generators x, y, z with $\deg x = 4$, $\deg y = 2m$, $\deg z = 2n$, and $p > n$, $m > 1$, then the integer m satisfies one of the following conditions* (A), *while n satisfies one of the following conditions* (B).

(A) $2 \equiv 1-p \bmod m$, (B) $2 \equiv 1-p \bmod n$.

$m \equiv 1-p \bmod m$, $n \equiv 1-p \bmod n$.

$n \equiv 1-p \bmod m$, $m \equiv 1-p \bmod n$.

Proof. Since $m \not\equiv 0 \bmod p$, it follows from Theorem 2.1 that one of the conditions (A) is satisfied. The remaining part of this corollary is proved similarly.

3. Euler function and its consequence. The next classical theorem is needed in this section.

Theorem (Dirichlet). *Every arithmetic series whose initial term and difference are relatively prime contains an infinite number of primes.*

Let φ be the Euler function. The elementary property of this function is as follows.

1) *If p is a prime, then $\varphi(p^e) = p^e - p^{e-1}$, $e \geqq 0$, where we understand $p^{-1} = 0$.*

2) *If a and b are relatively prime, then $\varphi(ab) = \varphi(a)\varphi(b)$.*

Lemma 3.1. If $n \neq 1, 2, 3, 4, 6$, *then* $\varphi(n) \geqq 4$.

Proof. If $n \neq 2^r 3^s$, ($r \geqq 0$, $s \geqq 0$), then n is the form of $t \cdot p^e$, $p \geqq 5$, $e \geqq 1$, $(t, p^e) = 1$.

Therefore

$$\begin{aligned} \varphi(n) &= \varphi(t)\varphi(p^e) = \varphi(t)(p^e - p^{e-1}) \\ &= \varphi(t)(p-1)p^{e-1} \geqq \varphi(t)(p-1) \\ &\geqq 4\varphi(t) \geqq 4. \end{aligned}$$

If n is the form $n = 2^r 3^s$ ($r \geqq 0$, $s \geqq 0$), then

$$\varphi(n) = (2^r - 2^{r-1})(3^s - 3^{s-1}).$$

By the assumption of this Lemma 3.1,

1) If $r=0$, then $s \geqq 2$.

2) If $r=1$, then $s \geqq 2$.

3) If $r=2$, then $s \geqq 1$.

4) If $r=3$, then $s \geqq 0$.
Therefore, we get $\varphi(n) \geqq 4$ in all these cases.

Lemma 3.2. *Let n be a positive integer. If $\varphi(n) \geqq 4$, then there is an integer e satisfying the following conditions*

$$\begin{aligned} &e \not\equiv -1 && \bmod n \\ &e \not\equiv 1 && \bmod n \\ &e \not\equiv 1-m && \bmod n \\ &(e, n)=1 && \end{aligned}$$

where m is a given integer.

Proof. Since $\varphi(n) \geqq 4$, there are relatively different (mod n) integers e_1, e_2, e_3, e_4, such that $(n, e_1)=1$, $(n, e_2)=1$, $(n, e_3)=1$, $(n, e_4)=1$. Therefore, if $e_{j_1} \equiv -1 \bmod n$, $e_{j_2} \equiv 1 \bmod n$, $e_{j_3} \equiv 1-m \bmod n$, then the remaining e_{j_4} is the required integer e. The other case is nothing to prove.

Proposition 3.3. *Let n and m be positive integers. If, for all sufficiently large prime p, one of the following conditions is satisfied, then $n=1, 2, 3, 4$, or 6.*

$$\begin{aligned} &p \equiv -1 && \bmod n \\ &p \equiv 1 && \bmod n \\ &p \equiv 1-m && \bmod n \end{aligned}$$

Proof. It $n \neq 1, 2, 3, 4, 6$, then, it follows from Lemma 3.1 that we get $\varphi(n) \geqq 4$. Therefore, from Lemma 3.2, there is an integer e such that $e \not\equiv -1 \bmod n$, $e \not\equiv 1 \bmod n$, $e \not\equiv 1-m \bmod n$, $(e, n)=1$. From Dirichlet's theorem, arithmetic series $\{a_j \mid a_j = e+nj\}$ contains infinitely many primes. Suppose that p is such a prime, then $p \equiv e \bmod n$.

Therefore

$$\begin{aligned} &p \not\equiv -1 && \bmod n \\ &p \not\equiv 1 && \bmod n \\ &p \not\equiv 1-m && \bmod n \end{aligned}$$

This concludes the proof.

It is easily checked by the same way as in [6] that these values 1, 2, 3, 4, 6 actually satisfy the above conditions for all sufficiently large prime p.

Proposition 3.4. *Let n and m be positive integers. If, for all sufficiently large prime p, one of the following conditions is satisfied, then $m=1, 2, 3, 4$, or 6.*

$$\begin{aligned} &p \equiv -1 && \bmod m \\ &p \equiv 1 && \bmod m \\ &p \equiv 1-n && \bmod m \end{aligned}$$

Proof. Since Lemma 3.1 and Lemma 3.2 are valid for m, the proof of this proposition is the same as Proposition 3.3. These values actually satisfy the above conditions for all sufficiently large prime

p also.

4. Proof of Theorem. In this section we will prove the main Theorem by the method of L. Smith [6]. Next Theorems are needed in this section.

Theorem 4.1 (L. Smith) [6]. *Let X be a connected associative H-space with $H_*(X;Z)$ finitely generated as an abelian group. If the rank of X is 2, then the type of X is either* (1, 1), (1, 3), (3, 3), (3, 5), (3, 7), *or* (3, 11).

Theorem 4.2 (A. Clark) [1]. *Let X be a simply connected associative H-space with $H_*(X;Z)$ finitely generated as an abelian group. Then $H_*(X;Z)$ has a generator of degree* 3.

Theorem 4.3. *Let X be an arcwise connected H-space and it's* mod p *cohomology $H^*(X;Z_p)$ is an exterior algebra on odd dimensional generators of rank* 3 *except the type* $(1, 1, *)$. *Then* mod p *cohomology $H^*(\bar{X};Z_p)$ of universal covering space $\bar{X}$ of X is*

$$H^*(\bar{X};Z_p)=\frac{H^*(X;Z_p)}{\{H^1(X;Z_p)\}}$$

where $\{H^1(X;Z_p)\}$ denotes the ideal generated by $H^1(X;Z_p)$. This is a corollary of W. Browder's Theorem [2].

Now, we are in a position to prove the main Theorem which is stated in the introduction.

Assume that X is an arcwise connected simply connected H-space of rank 3 wiht $H_*(X;Z)$ finitely generated as an abelian group. Then it follows from Hopf's classical theorem and Theorem 4.2 that

$$H^*(X;Q)=E[x, y, z]$$

where $\deg x=3$, $\deg y=2m-1$, $\deg z=2n-1$. $H_*(X;Z)$ is finitely generated and therefore has torsion for only a finite number of primes. Dold and Lashof [4] have shown that an associative H-space X has the classifying space BX.

Then

$$H^*(BX;Z_p)\cong P[u, v, w]$$

where $\deg u=4$, $\deg v=2m$, $\deg w=2n$, for all sufficiently large prime p.

This generalization of the Borel transgression the theorem is justified by the paper [4] and [5], and is found in the paper [1], [6]. From Corollary 2.2, it follows that for all sufficiently large prime p, m satisfies one of the condition (A), and n satisfies one of the condition (B) in that corollary. Then we get from Proposition 3.3 and Proposition 3.4 that $n=1, 2, 3, 4$, or 6, $m=1, 2, 3, 4$, or 6.

Since we assumed that X is simply connected, the case $n=1$ is excluded. Thus the possible types of X are (3, 3, 3), (3, 3, 5), (3, 3, 7), (3, 3, 11), (3, 5, 5), (3, 5, 7), (3, 5, 11), (3, 7, 7), (3, 7, 11), (3, 11, 11). If

X is not simply connected and does not have the type $(1, 1, *)$, then we apply Theorem 4.1 (L. Smith) to the universal covering space $\bar{X}$ of X together with Theorem 4.3 and we get that type of X is either $(1, 3, 3)$, $(1, 3, 5)$, $(1, 3, 7)$, or $(1, 3, 11)$. If the type is $(1, 1, *)$, then by Theorem 2.1, we obtain easily such a type is either $(1, 1, 1)$, or $(1, 1, 3)$. Thus all the possible types of X are $(1, 1, 1)$, $(1, 1, 3)$, $(1, 3, 3)$, $(1, 3, 5)$, $(1, 3, 7)$, $(1, 3, 11)$, $(3, 3, 3)$, $(3, 3, 5)$, $(3, 3, 7)$, $(3, 3, 11)$, $(3, 5, 7)$, $(3, 7, 11)$, $(3, 5, 5)$, $(3, 5, 11)$, $(3, 7, 7)$, $(3, 11, 11)$.

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