175. On the Sets of Points in the Ranked Space. II

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In the preceding papers [3], [4], we defined the notions with respect to sets of points, m-open, m-closed, m-accumulation point, m-ω-accumulation point, m-cluster point, in the ranked space [1], [2]. And, we mentioned some propositions in respect of these notions. However, in these propositions, we assumed that a ranked space satisfies Condition (M).

Now, we find that all of these propositions hold in the ranked space which does not satisfy Condition (M), if we use new notions derived by another definition of an open set.

1. Definition 1. A subset A in a ranked space is r-open if and only if for any point p belonging to A and for any fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, there is some member $V_{\alpha_0}(p)$ of $\{V_\alpha(p)\}$ such that $V_{\alpha_0}(p)\subseteq A$.

A subset B in a ranked space is r-closed if and only if the complementary set of B, $R-B$, is r-open.

Definition 2. In a ranked space R, a point p is an r-accumulation point of a subset A if and only if there is some fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $V_\alpha(p)\cap(A-\{p\})\neq\phi$ for all α.

Proposition 1. *If R is a ranked space, then the following conditions are equivalent.*

(a) *A subset A of R is r-closed.*

(b) *A subset A of R contains the set consisting of its r-accumulation points.*

Proof. We can prove this proposition as well as the proposition in the preceding paper [3].

Proposition 2. *If R is a ranked space, then the following conditions are equivalent.*

(a) *A point p is an r-accumulation point of a subset A of R.*

(b) *There is a sequence in $A-\{p\}$ which R-converges to p.*

Proof. To prove that (a) implies (b).

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We can prove this as well as the proposition in the preceding paper [3].

To prove that (b) implies (a).

Let $\{p_\alpha\}$ be a sequence in $A-\{p\}$ which R-converges to p. Then there is a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $p_\alpha \in V_\alpha(p)$ and $p_\alpha \in A-\{p\}$. Consequently p is an r-accumulation point of A.

Proposition 3. *If R is a ranked space, then the following conditions are equivalent.*

(a) *Each sequence which R-converges to a point of a subset A of R is eventually in A.*

(b) *A subset A of R is r-open.*

Proof. To prove that (a) implies (b).

We can prove this as well as the proposition in the preceding paper [3].

To prove that (b) implies (a).

Let $\{p_\alpha\}$ be a sequence which R-converges to a point p of A. Then there is a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $p_\alpha \in V_\alpha(p)$. Since A is r-open and $p \in A$, there is a member $V_{\alpha_0}(p)$ of $\{V_\alpha(p)\}$ such that $V_{\alpha_0}(p) \subseteq A$. Consequently the sequence $\{p_\alpha\}$ is eventually in A.

Definition 3. A point p is an r-ω-accumulation point of a subset A of a ranked space if and only if there is some fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that each number $V_\alpha(p)$ of $\{V_\alpha(p)\}$ contains infinitely many points of A.

Definition 4. A point p is an r-cluster point of a sequence $\{p_\alpha\}$ in a ranked space if and only if there is some fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $\{p_\alpha\}$ is frequently in each member $V_\alpha(p)$ of $\{V_\alpha(p)\}$.

Proposition 4. *If R is a ranked space, then the following conditions are equivalent.*

(a) *Every infinite subset of R has an r-ω-accumulation point.*

(b) *Every sequence in R has an r-cluster point.*

(c) *For every sequence in R there is a subsequence R-converging to a point of R.*

Proof. To prove that (c) implies (b).

Let $\{p_\beta\}$ be an arbitrary sequence of R. Since $\{P_\beta\}$ has a subsequence $\{p_{\beta_\alpha}\}$ R-converging to a point p of R, there is a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $p_{\beta_\alpha} \in V_\alpha(p)$. Consequently the point p is an r-cluster point of the sequence $\{p_\beta\}$.

To prove that (b) implies (a).

We can prove this as well as the proposition in the preceding paper [3].

To prove that (a) implies (c).

Let $\{p_\alpha\}$ be an arbitrary sequence of R. In the case where the range of the sequence is infinite, by the condition (a), the sequence $\{p_\alpha\}$ has an r-ω-accumulation point p. Hence there is some fundamental sequence of neighborhoods of p, $\{V_\beta(p)\}$, such that each member $V_\beta(p)$ of $\{V_\beta(p)\}$ contains infinitely many points of the sequence $\{p_\alpha\}$. Consequently we can choose a subsequence $\{p_{\alpha_\beta}\}$ from the sequence $\{p_\alpha\}$, such that $p_{\alpha_\beta} \in V_\beta(p)$ and $p_{\alpha_\beta} \neq p_{\alpha_\delta}$ for $\delta = 1, 2, \cdots, \beta - 1$. The sequence $\{p_{\alpha_\beta}\}$ is a subsequence of $\{p_\alpha\}$ and it R-converges to the point p of R. In the case where the range of the sequence is finite, we have $p_{\alpha_\beta} = p$ for some point p of the ranked space, for infinitely many natural numbers α_β. Consequently the sequence $\{p_{\alpha_\beta}\}$ is a subsequence of $\{p_\alpha\}$ and it R-converges to the point p of R.

2. Moreover, if the content of the statement that the axiom (T_2) of separetion is satisfied in the ranked space is defined as below, we can show that following propositions hold in the ranked space which does not satisfy Condition (M).

We say that a ranked space R satisfies the axiom (T_2') of separation if and only if for distinct points p and q of R, and for any fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, and of q, $\{U_\alpha(q)\}$, there exist disjoint members of $\{V_\alpha(p)\}$ and $\{U_\alpha(q)\}$, that is, $V_\alpha(p) \cap U_\alpha(q) = \phi$ for some α.

Proposition 1. *Let $\{p_\alpha\}$ be an R-convergent sequence of a ranked space R. Then the following conditions are equivalent.*

(a) *R satisfies the axiom (T_2') of separation.*

(b) *$\{\lim_\alpha p_\alpha\}$ consists of the only point.*

Proof. To prove that (a) implies (b).

Suppose that $p, q \in \{\lim_\alpha p_\alpha\}$ and $p \neq q$. Since $p, q \in \{\lim_\alpha p_\alpha\}$, there exists a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $p_\alpha \in V_\alpha(p)$, and a fundamental sequence of neighborhoods of q, $\{U_\alpha(q)\}$, such that $p_\alpha \in U_\alpha(q)$. Consequently, $p_\alpha \in V_\alpha(p) \cap U_\alpha(q)$ for all α. This result contradicts the fact that R satisfies the axiom (T_2') of separation.

To prove that (b) implies (a).

Suppose that R does not satisfy the axiom (T_2') of separation. Then, for any distinct points p and q of R, there exists a pair of fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, and of q, $\{U_\alpha(q)\}$, such that $V_\alpha(p) \cap U_\alpha(q) \neq \phi$. Hence we can choose a point p_α belonging to $V_\alpha(p) \cap U_\alpha(q)$ for all α. The sequence $\{p_\alpha\}$ is R-convergent to p and to q. This is a contradiction.

Proposition 2. *Let R be a ranked space satisfying the axiom (T_2') of separation. If a subset A of R is sequentially compact and $\{p_\alpha\}$ is an R-convergent sequence of A, then* $\{\lim_\alpha p_\alpha\} \subseteq A$.

Proof. We can prove this proposition as well as the proposition in the preceding paper [4].

Proposition 3. *Let R be a ranked space satisfying the axiom (T_2') of separation. If a subset A of R is sequentially compact, then A is r-closed.*

Proof. Suppose that A is not r-closed. Since $R-A$ is not r-open, there exist some point x_0 belonging to $R-A$ and some fundamental sequence of neighborhoods of x_0, $\{V_\alpha(x_0)\}$, such that $V_\alpha(x_0) \cap A \neq \phi$ for all α. Hence we can choose a point p_α belonging to $V_\alpha(x_0) \cap A$ for all α. Then, the sequence $\{p_\alpha\}$ is contained in A and it is R-convergent to x_0 belonging to $R-A$. On the other hand, by Proposition 2, we have $x_0 \in A$. This is a contradiction.

Proposition 4. *Let R be a sequentially compact ranked space. If a subset A of R is r-closed, then A is sequentially compact.*

We can prove this proposition as well as the proposition in the preceding paper [4].

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