169. On Lacunary Trigonometric Series. II

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§ 1. Introduction. In [3] we have proved

Theorem A. Let \( \{n_k\} \) be a sequence of positive integers and \( \{a_k\} \) a sequence of non-negative real numbers for which the conditions

\[
\begin{align*}
    n_{k+1} &> n_k(1 + ck^{-\alpha}), \quad k = 1, 2, \ldots, \\
    A_N &= (2^{-1} \sum_{k=1}^{N} a_k^{1/2})^{1/2} \to + \infty, \quad \text{as } N \to + \infty,
\end{align*}
\]

and

\[
    a_N = o(A_N N^{-\alpha}), \quad \text{as } N \to + \infty,
\]

are satisfied, where \( c \) and \( \alpha \) are any given constants such that

\[
    c > 0 \quad \text{and} \quad 0 \leq \alpha \leq 1/2.
\]

Then we have, for all \( x \),

\[
    \lim_{N \to \infty} \left| \frac{1}{|E|} \sum_{k=1}^{N} a_k \cos 2\pi n_k (t + \alpha_k) \right| = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-u^2/2)du, \quad (*)
\]

where \( E \subset [0, 1] \) is any given set of positive measure and \( \{\alpha_k\} \) any given sequence of real numbers.

This theorem was first proved by R. Salem and A. Zygmund in case of \( \alpha = 0 \), where \( \{n_k\} \) satisfies the so-called Hadamard's gap condition (cf. [4], (5.5), pp. 264–268). In that case they also remarked that under the hypothesis (1.2) the condition (1.3) is necessary for the validity of (1.5) (cf. [4], (5.27), pp. 268–269).

Further, in [2] P. Erdős has pointed out that for every positive constant \( c \) there exists a sequence of positive integers \( \{n_k\} \) such that \( n_{k+1} > n_k(1 + ck^{-1/2}), \ k \geq 1 \), and (1.5) is not true for \( a_k = 1, \ k \geq 1 \), and \( E = [0, 1] \). But I could not follow his argument on the example.

The purpose of the present note is to prove the following

Theorem B. For any given constants \( c > 0 \) and \( 0 \leq \alpha \leq 1/2 \), there exist sequences of positive integers \( \{n_k\} \) and non-negative real numbers \( \{a_k\} \) for which the conditions (1.1), (1.2) and

\[
    a_N = O(A_N N^{-\alpha}), \quad \text{as } N \to + \infty,
\]

are satisfied, but (1.5) is not true for \( E = [0, 1] \) and \( \alpha_k = 0, \ k \geq 1 \).

The above theorem shows that in Theorem A the condition (1.3) is

\* \* \* |E| denotes the Lebesgue measure of a set E.
indispensable for the validity of (1.5). In §§3–5 we prove Theorem B for $0<\alpha\leq1/2$.

§2. Some lemmas. i. In this section let $\{X_k(\omega)\}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, P)$ with vanishing mean values and finite variances. Putting $E(X_k^2)=\sigma_k^2$ and $s_m^2=\sum_{k=1}^m \sigma_k^2$, the theorem of Lindeberg reads as follows:

Theorem. (Cf. [1] pp. 56–57.) Under the hypotheses
\begin{equation}
\sigma_m=0(s_m), \text{ as } m\to\infty, \tag{2.1}
\end{equation}
a necessary and sufficient condition for the validity of the relation
\begin{equation}
\lim_{m\to\infty} P\left\{s_m^{-1} \sum_{k=1}^m X_k(\omega) \leq x\right\}=(2\pi)^{-1/2} \int_{-\infty}^x \exp(-u^2/2)du \tag{2.2}
\end{equation}
for all $x$ is that, for any given $\varepsilon>0$,
\begin{equation}
\lim_{m\to\infty} s_m^{-2} \sum_{k=1}^m \int_{X_k(\omega)>s_m} X_k^2(\omega) dP(\omega)=0. \tag{2.3}
\end{equation}
From the above theorem the following lemma is easily seen.

Lemma 1. Under the hypotheses (2.1), the relation (2.2) implies that, for any given $\varepsilon>0$,
\begin{equation}
\lim_{m\to\infty} \sum_{k=1}^m P\{X_k(\omega) > \varepsilon s_m\}=0. \tag{2.3}
\end{equation}

ii. Lemma 2. Let $m$ and $l$ be any given positive integers. Then there exists a positive constant $c_0$, not depending on $m$ and $l$, such that
\begin{align*}
|\{t; t \in [0, 1], |\sum_{j=0}^l \cos 2\pi m(j+1)t| > (l+1)/3\}| \geq 2c_0 l^{-1}.
\end{align*}

Proof. This can be easily seen from the relation
\begin{align*}
\sum_{j=0}^l \cos 2\pi m(j+1)t &= \frac{\sin 2\pi m(l+3/2)t}{2 \sin \pi mt} - 1/2,
\end{align*}
provided if $\sin \pi mt \neq 0$.

§3. Construction of sequences. In the following let $c>0$ and $0<\alpha\leq1/2$ be given constants in Theorem B. First let us put
\begin{align*}
p(j)=\lfloor j^{1/\alpha}\rfloor, \quad l(j)=\operatorname{Min}\{[p^\alpha(j)/c], p(j+1)-p(j)-1\}, \quad j_0=\operatorname{Min}\{j; l(j)\geq1\}. \tag{3.1}
\end{align*}
Since $p(j+1)-p(j)\sim \alpha^{-1}j^{(1-\alpha)/\alpha}$ and $p^\alpha(j)\sim j$, as $j\to+\infty$, we have
\begin{equation}
l(j)\sim \beta(\alpha)j, \quad \text{as } j\to+\infty, \tag{3.2}
\end{equation}
where
\begin{equation}
\beta(\alpha) = \begin{cases} 1/c, & \text{if } 0<\alpha<1/2, \\ \operatorname{Min}(2, 1/c), & \text{if } \alpha=1/2. \end{cases} \tag{3.3}
\end{equation}
Next we put
\begin{align*}
n_1=1 \quad \text{and} \quad n_{k+1}=[n_k(1+ck^{-\alpha})+1], \quad \text{for } k+1<p(j_0).
\end{align*}
If $n_{p(j)}$, $j\geq j_0$, is defined, then we put
\begin{align*}
\text{[x]} & \text{ denotes the integral part of } x, \\
\text{[4]} & \text{ f(j)~g(j), as j~+\infty, means that } f(j)/g(j)\to1, \text{ as } j\to+\infty.
\end{align*}
Further we put, for \( j \geq j_0 \),

\[ n_{p(j)} = 2^{q(j)}, \]

where

\[
q(j) = \begin{cases} 
\min \{ m \colon 2^m > \frac{n_{p(j-1)}}{n_{p(j)}}, \frac{n_{p(j-1)}}{n_{p(j)} + 1} \} + 1, & \text{if } j < j_0, \\
\min \{ m \colon 2^m > \frac{n_{p(j-1)}}{n_{p(j)}}, \frac{n_{p(j-1)}}{n_{p(j)} + 1} \} + 1, & \text{if } j > j_0.
\end{cases}
\]

Then it is clear that the sequence \( \{ n_k \} \) satisfies (1.1).

On the other hand we define \( \{ a_k \} \) as follows:

\[
a_k = \begin{cases} 
1, & \text{if } p(j) \leq k \leq p(j) + l(j), \text{ for some } j \geq j_0, \\
\frac{1}{k^2}, & \text{if otherwise.}
\end{cases}
\]

Then we have, by (3.6) and (3.2),

\[
A_{2\lambda(x)} = 2^{-1} \sum_{m=0}^{\infty} \{ (l(j) + 1) + O(1) \} \beta(x) m^2 / 4, \quad \text{as } m \to \infty.
\]

Since \( p^*(j) \sim j \), as \( j \to \infty \), this sequence \( \{ a_k \} \) satisfies both of the conditions (1.2) and (1.6).

Further, if we put \( S_N(t) = \sum_{k=1}^{N} a_k \cos 2\pi n_k t \), then we have, by the definitions of \( \{ n_k \} \) and \( \{ a_k \} \), uniformly in \( t \),

\[
S_{p(j) + l(j)}(t) = \sum_{m=0}^{\infty} \cos \{ 2\pi 2^{q(j)}(l+1)t \} + O(1), \quad \text{as } m \to \infty.
\]

§ 4. Independent functions. Let \( x_k(t) \) be the \( k \)-th digit of the infinite dyadic expansion of \( t \), \( 0 \leq t \leq 1 \), that is,

\[
t = \sum_{k=1}^{\infty} x_k(t) 2^{-k}, \quad (x_k(t) = 0 \text{ or } 1),
\]

then \( \{ x_k(t) \} \) is a sequence of independent functions on the interval \([0, 1] \). Putting

\[
\eta_j(t) = \sum_{k=q(j)}^{q(j+1)-1} x_k(t) 2^{-k}, \quad \text{for } j \geq j_0,
\]

we define

\[
\mu_j = \int_0^{1} \sum_{l=0}^{1} \cos 2\pi 2^{q(j)}(l+1) \eta_j(t) dt,
\]

\[
Y_j(t) = \sum_{l=0}^{1} \cos \{ 2\pi 2^{q(j)}(l+1) \eta_j(t) \} - \mu_j,
\]

\[
\tau_j = \int_0^{1} Y_j(t) dt \quad \text{and} \quad t^n_m = \sum_{j=j_0}^{m} \tau_j.
\]

Then we have, by (4.2) and (3.5),

\[
\sup \left| \sum_{l=0}^{1} \cos \{ 2\pi 2^{q(j)}(l+1) \eta_j(t) \} - \sum_{l=0}^{1} \cos 2\pi \{ 2^{q(j)}(l+1) \eta_j(t) \} \right|
\]

\[
= O(\sup_{t} 2^{q(j)} \sum_{k=q(j)+1}^{\infty} x_k(t) 2^{-k} \sum_{l=0}^{1} (l+1))
\]

\[
= O(2^{q(j)} j^2 \sum_{k=q(j)+1}^{\infty} 2^{-k})
\]

\[
= O(2^{q(j)-q(j+1)} j^2) = O(j^{-1}), \quad \text{as } j \to \infty.
\]
Therefore, we have
\[(4.4) \sup_{t} |Y_j(t) - \sum_{l=0}^{(j-1)/2} \cos 2\pi 2^q(j)(l+1)t| = O(j^{-1}), \] as \(j \to +\infty\),
and, by (3.2) and (3.7),
\[(4.5) \quad t^2_m = 2^{-1} \sum_{j=0}^{m} (l(j)+1)^2 + O(\log m)\]
\[\sim A^2_{p(m)+1(m)} \sim \beta(\alpha)m^3/4, \] as \(m \to +\infty\).
Thus we obtain
\[(4.6) \quad t_m \to +\infty \quad \text{and} \quad \tau_m = o(t_m), \] as \(m \to +\infty\).
Further, we have, by (3.8) and (4.4),
\[(4.7) \quad |S_{p(m)+1(m)}(t) - \sum_{j=0}^{m} Y_j(t)| = O(\log m)\]
\[\quad = o(A^2_{p(m)+1(m)}), \] uniformly in \(t\), as \(m \to +\infty\).
Since (4.5) and (3.2) imply that \(\sqrt{\beta(\alpha)} t_m/5 < [l([m/2]) + 1]/4\), for \(m \geq m_0\),
we have, by (4.4)
\[\sum_{j=m/2}^{m} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. \left| Y_j(t) \right| > \sqrt{\beta(\alpha)} t_m/5 \right\} \right| \]
\[\geq \sum_{j=m/2}^{m} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. \sum_{i=0}^{(j-1)/2} \cos 2\pi 2^q(j)(l+1)t > [l(j)+1]/3 \right\} \right| \]
and by Lemma 2, we have
\[(4.8) \quad \lim_{m \to +\infty} \sum_{j=m/2}^{m} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. \left| Y_j(t) \right| > \sqrt{\beta(\alpha)} t_m/5 \right\} \right| \]
\[\geq \lim_{m \to +\infty} 2c_0 \sum_{j=m/2}^{m} \left| l^{-1}(j) \right| \geq \lim_{m \to +\infty} c_0 \beta(\alpha)^{-1} \sum_{j=m/2}^{m} j^{-1} > 0. \]

§ 5. Proof of Theorem B. Suppose that (1.5) holds, that is,
\[(5.1) \quad \lim_{m \to +\infty} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. A^{-1}_{p(m)+1(m)} S_{p(m)+1(m)}(t) \leq x \right\} \right| \]
\[= (2\pi)^{-1/2} \int_{-\infty}^{x} \exp \left( -u^2/2 \right) du. \]
Then by (4.7) and (4.5), we have
\[(5.2) \quad \lim_{m \to +\infty} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. t_m^{-1} \sum_{j=0}^{m} Y_j(t) \leq x \right\} \right| \]
\[= (2\pi)^{-1/2} \int_{-\infty}^{x} \exp \left( -u^2/2 \right) du. \]
On the other hand (4.1), (4.2), and (4.3) imply that \(\{Y_j(t)\}\) is a sequence
of independent functions with vanishing mean values and finite variances.
By (5.2) and (4.6) we can apply Lemma 1 to \(\{Y_j(t)\}\) and obtain
\[\lim_{m \to +\infty} \sum_{j=m/2}^{m} \left| \left\{ t \mid 0 \leq t \leq 1, \right. \right. \]
\[\left. \left. Y_j(t) > \sqrt{\beta(\alpha)} t_m/5 \right\} \right| = 0. \]
This contradicts with (4.8).

References

Tract (1962).
(1959).