166. *On Semi-Groups in Banach Algebras Close to the Identity*

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In their recent paper [1], Nakamura and Yoshida showed that the identity I is the only bounded operator T on a Hilbert space which satisfies $||T^n - I|| \leq \delta < 1$ for $n = 1, 2, 3, \cdots$. Their proof is based on the mean ergodic theorem.

In the present note a more general result is derived by means of a Bourbaki exercise on spectral theory in Banach algebras.

Theorem. *Let A be a complex Banach algebra with an identity e and $S \subset A$ a multiplicative semi-group* (not necessarily commutative) *such that $||s - e|| \leq \delta < 1$ for every $s \in S$. Then, $S = \{e\}$.*

Proof. Let s be an arbitrary element of S, so that $||s^n - e|| \leq \delta < 1$ for $n \in IN$. We have (s clearly being invertible)

(1) $\qquad ||s^n|| \leq 1 + \delta$ and $||s^{-n}|| \leq (1 - \delta)^{-1}, \qquad n \in IN.$

The first statement being obvious, the second follows from

$$||s^{-n}|| \leq ||s^{-n} - e|| + 1 = ||s^{-n}(e - s^n)|| + 1 \leq ||s^{-n}|| \delta + 1.$$

Next, if $\lambda \in \sigma(s)$, then $\lambda^n - 1 \in \sigma(s^n - e)$. Simple arguments make it plain that the ensuing inequalities $|\lambda^n - 1| \leq ||s^n - e|| \leq \delta$; $n \in IN$, have the unique solution $\lambda = 1$. Consequently, $q = s - e$ has spectrum $\{0\}$.

Finally, we invoke [2], p. 92, Exerc. 24 b): if q is quasi-nilpotent in $s = e + q$, then $q^k = 0$ for some $k \in IN$ is equivalent to $\lim_{n \to \infty} n^{-k} ||s^{\pm n}|| = 0$. It follows from (1) that this limit is zero for $k = 1$, whence $q = 0$ and we are done.

Remark. The example $A = C[0, 1]$, $S = \{s \in A \cdot 0 < s \leq 1\}$ shows that the theorem breaks down upon weakening the assumption to $||s - e|| < 1$ for $s \in S$.

References

[1] M. Nakamura and M. Yoshida: On a generalization of a theorem of Cox. Proc. Japan Acad., **43**, 108–110 (1967).
[2] N. Bourbaki: Théories spectrales. Chap. I et II, Act. Sci. Ind., No. 1332, Hermann, Paris (1967).