202. An Asymptotic Property of Gaussian Stationary Processes

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Let $X=\{x(t), -\infty<t<\infty\}$ be a real separable stochastically continuous Gaussian stationary process defined on a probability measure space $(\Omega, \mathscr{B}, P)$. We assume that $E(x(t))=0$ and $E(x^2(t))=1$. We put $r(t)=E(x(t)x(0))$ and $\sigma^2(h)=E((x(t+h)-x(t))^2)$.

If the sample functions are almost certainly everywhere continuous, for every fixed $T>0$, the quantity

$$\eta(T)=\max_{0\leqq t\leqq T} x(t)$$

will have a definite meaning. In this note, we announce some results on the asymptotic behaviour of the processes $\{x(t), -\infty<t<\infty\}$ and $\{\eta(t), -\infty<t<\infty\}$.

We introduce the following conditions:

A, 1) There are constants C_1, δ_1 such that

$$\sigma^2(h)\leqq C_1 h^\alpha$$

for all h in $(0, \delta_1)$ for some α with $0<\alpha\leqq 2$.

A, 1′) There are constants C_2, δ_2 such that

$$C_2 h^\alpha \leqq \sigma^2(h)$$

for all h in $(0, \delta_2)$ for some α with $0<\alpha\leqq 2$.

A, 2) $\operatorname*{Lim\,sup}_{t\to\infty} r(t)\log t\leqq 0$.

Theorem 1. *Suppose that the condition* A, 1) *is satisfied and that* $\sigma^2(h)$ *is monotone non-decreasing in* $(0, \delta_1)$. *Let* $\varphi(t)$ *be a monotone non-decreasing continuous function for large* t's. *If*

$$\int^\infty \varphi(t)^{\frac{2}{\alpha}-1}\exp\left(-\frac{1}{2}\varphi^2(t)\right)dt<+\infty,$$

then we have

$$P(\text{there is a } t_0(\omega) \text{ such that } x(t)\leqq\varphi(t) \text{ for all } t\geqq t_0)=1$$

or equivalently

$$P(\text{there is a } T_0(\omega) \text{ such that } \eta(T)\leqq\varphi(T) \text{ for all } T\geqq T_0)=1.$$

Theorem 2. *Suppose that conditions* A, 1′) *and* A, 2) *are satisfied and that* $\sigma^2(h)$ *is monotone non-decreasing function in* $(0, \sigma_2)$. *Let* $\varphi(t)$ *be a monotone non-decreasing function for large* t's.

If

$$\int^\infty \varphi(t)^{\frac{2}{\alpha}-1}\exp\left(-\frac{1}{2}\varphi^2(t)\right)dt=\infty,$$

then we have

P(*for every* $t>0$, *there is a* $t_0(\omega)$ *such that* $x(t_0)>\varphi(t_0)$, $t_0(\omega)>t)=1$
or equivalently

P(*for every* $T>0$, *there is a* $T_0(\omega)$ *such that*
$\eta(T_0)>\varphi(T_0)$, $T_0(\omega)>T)=1$.

Combining Theorems 1 and 2, we have

Theorem 3. *Assume that conditions* (A, 1), (A, 1′), *and* (A, 2) *are satisfied at the same time for some* $\delta=\delta_1=\delta_2$ *and some* α *and that* $\sigma^2(h)$ *is monotone non-decreasing in* $(0, \delta)$. *Let* $\varphi(t)$ *be a monotone non-decreasing function for large* t's. *Then,*

P(*there is a* $t_0(\omega)$ *such that* $x(t)\leqq\varphi(t)$ *for all* $t\geqq t_0)=1$ *or* 0 *according as the integral*

$$\int^{\infty}\varphi(t)^{\frac{2}{\alpha}-1}\exp\left(-\frac{1}{2}\varphi^2(t)\right)dt$$

converges or diverges.

Corollary 1. *Under the same conditions as in Theorem* 3, *we have, for every* $\varepsilon>0$,

P(*there is a* $t_0(\omega)$ *such that*

$$x(t)\leqq\left(2\log t+\left(\frac{2}{\alpha}+1+\varepsilon\right)\operatorname{loglog} t\right)^{\frac{1}{2}} \text{ for all } t\geqq t_0)=1.$$

Moreover we have, for any $\varepsilon\geqq 0$,

P(*there is a* $t_0(\omega)$ *such that*

$$x(t)\leqq\left(2\log t+\left(\frac{2}{\alpha}+1-\varepsilon\right)\operatorname{loglog} t\right)^{\frac{1}{2}})=0.$$

From Corollary 1, it follows that, for every $\varepsilon>0$,

P(*there is a* $T_0(\omega)$ *such that*

$$\eta(T)\leqq\sqrt{2\log T}+\frac{\left(\frac{1}{\alpha}+\frac{1}{2}+\varepsilon\right)\operatorname{loglog} T}{\sqrt{2\log T}} \text{ for all } T\geqq T_0)=1. \tag{1}$$

H. Cramér [1] and M. G. Shur [3] have obtained the results corresponding to (1), in the case where $\alpha=2$. Assumptions A, 1), A, 1′), and A, 2) are almost equivalent to those introduced in Theorem 5, 4 of J. Pickands III. [2].

References

[1] H. Cramér: On the maximum of a normal stationary stochastic process. Bull. Amer. Math. Soc., **68**, 512–516 (1962).

[2] J. Pickands III.: Maxima of stationary Gaussian processes. Z. Wahrscheinlichkeitstheorie verw. Geb., **7**, 190–223 (1967).

[3] M. G. Shur: On the maximum of a Gaussian stationary process. Theor. Probab. Appl., **10**, 354–357 (1965).