200. An Extension of Wild's Sum for Solving Certain Non-linear Equation of Measures

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1. Let S be a compact space with the second countability axiom. Let $\mathfrak{W}$ be the set of all signed measures on the topological Borel field of S with finite total variation, and let $\mathfrak{W}_s$ (resp. $\mathfrak{W}_p$) be the subset of $\mathfrak{W}$ of all substochastic (resp. probability) measures. In $\mathfrak{W}$, we introduce the topology of weak convergence. Consider a non-linear equation:

$$\frac{du(t)}{dt}=B[u(t)]-u(t), \qquad u(0+)=f,^{1)} \tag{1}$$

where the initial value f and the solution $u(t)$ are in $\mathfrak{W}_s$ and $B[u]$ is given by the formula:

$$B[u]=\sum_{n=1}^{\infty} a_n B_n[u, \cdots, u], \tag{2}$$

for given $\{a_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ such that i) a_n is a non-negative real number, $a_1<1$ and $\sum_{n=1}^{\infty} a_n=1$, ii) B_n is a mapping from $\mathfrak{W}^n$ to $\mathfrak{W}$, multilinear, continuous and maps $\mathfrak{W}_p^n$ into $\mathfrak{W}_p$, for each $n\geqq 1$, where $\mathfrak{W}^n$ and $\mathfrak{W}_p^n$ mean the n-fold direct products of the spaces $\mathfrak{W}$ and $\mathfrak{W}_p$ respectively. This equation was considered by H. Tanaka [6] and T. Ueno [7], in a slightly different form, to extend the result of McKean [5] and Johnson [4] concerning the propagation of chaos. In [6], the following condition:

$$\int_{1-\varepsilon}^{1} \frac{d\xi}{\xi-\sum_{n=1}^{\infty} a_n \xi^n}=+\infty \qquad \text{for any} \quad \varepsilon>0, \tag{3}$$

is assumed to prove the propagation of chaos. This condition seems closely related to the condition of the uniqueness of the solution of (1). In this paper, as a remark to [6], we give an extension of Wild's sum for the solution of the equation (1) and investigate the relation between the condition (3) and the uniqueness of the solution of (1).

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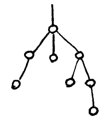
1) In this paper, the continuity, differentiability and integral of $u(t)$ are in the sense of topology of weak convergence in $\mathfrak{M}$. In equation (1), we assume the differentiability of $u(t)$ as a matter of course.

2. To give an extension of Wild's sum [7], we define the set of trees inductively in the following manner. T_1 is the set of only one element (α) where α is an symbol called "surmit". If T_k is defined for each $k \leqq n$, then T_{n+1} is defined by:

$$(4)\quad T_{n+1}=\bigcup_{m=1}^{\infty} \bigcup_{\substack{1 \leqq n_1, \cdots, n_m \leqq n \\ n_1+\cdots+n_m=n}} \{p=(p_1, \cdots, p_m)\,;\, p_1 \in T_{n_1}, \cdots, p_m \in T_{n_m}\},$$

and $T=\bigcup_{n=1}^{\infty} T_n$, where unions mean direct sum. This notation is one representation of the branching trees as in the following:

$$T_8 \ni (\underbrace{((\alpha))}_{p_1}, \underbrace{(\alpha)}_{p_2}, \underbrace{((\alpha), ((\alpha)))}_{p_3}) \leftrightarrow$$



Because of this definition, we can "inductively" define any notation $N(p)$ depending on $p \in T$, if we define initially $N((\alpha))$ and then define $N(p)$ by means of $N(p_1), \cdots, N(p_m)$ in the case $p=(p_1, \cdots, p_m)$. So, define $f^p \in \mathfrak{W}_s$ for $f \in \mathfrak{W}_s$ and $G^p(t) \in C([0, \infty))$ in the following:

$$(5)\quad \begin{cases} f^{(\alpha)}=f, \\ f^p=B_m[f^{p_1}, \cdots, f^{p_m}], \qquad \text{if} \quad p=(p_1, \cdots, p_m), \end{cases}$$

$$(6)\quad \begin{cases} G^{(\alpha)}(t)=e^{-t}, \\ G^p(t)=a_m\displaystyle\int_0^t e^{-(t-s)} G^{p_1}(s) \cdots G^{p_m}(s) ds, \qquad \text{if} \quad p=(p_1, \cdots, p_m). \end{cases}$$

The convergence of the right hand side of (6) is easily seen if we prove inductively that $0 \leqq G^p(t) \leqq 1$. Then we can define an extension of Wild's sum:

$$(7)\quad u(t, f)=\sum_{p \in T} G^p(t) f^p.$$

3. For $u, v \in \mathfrak{W}$, we define $u \geqq v$ if and only if $u=v+w$ for some non-negative measure w. Then we have

Theorem 1. (a) *$u(t, f)$ is the minimal solution of* (1). (b) *If $f(S)<1$, then* (1) *has unique solution* (c). *If the condition* (3) *is satisfied, then* (1) *has unique solution for any $f \in \mathfrak{W}_p$.* (d) *If the condition* (3) *is not satisfied, then the solution of* (1) *is not unique for some $f_0 \in \mathfrak{W}_p$.*

Before proving this theorem, we prove the following three lemmas:

Lemma 1. (1) *is equivalent to the following integral equation:*

$$(8)\quad u(t)=e^{-t} f+\int_0^t e^{-(t-s)} B[u(s)] ds.^{2)}$$

Proof. Let (8) be satisfied. The differentiability of $u(t)$ is clear by the right hand side of (8), and differentiating (8) in t we get (1). Conversely let (1) be satisfied. The continuity of $u(t)$ is clear, and integrating by parts we have

2) In equation (8), we assume the continuity of $u(t)$.

$$\int_0^t e^{-(t-s)}u'(s)ds = u(t) - e^{-t}f - \int_0^t e^{-(t-s)}u(s)ds,$$

so we get (8).

Lemma 2. *Let $u_n(t)$ be a sequence of successive approximation of* (8):

$$(9) \qquad \begin{aligned} &u_0(t) = e^{-t}f \\ &u_{n+1}(t) = e^{-t}f + \int_0^t e^{-(t-s)}B[u_n(s)]ds, \qquad \text{for} \quad n \geqq 0. \end{aligned}$$

Then $u_u(t)$ increases to $u(t, f)$, as $n \uparrow +\infty$.

Proof. Define the length $L(p)$ of the tree $p \in T$ by $L((\alpha)) = 0$ and $L(p) = \max_{1 \leqq i \leqq m} L(p_i) + 1$ if $p = (p_1, \cdots, p_m)$, and let $T(n) = \{p\,;\, L(p) \leqq n\}$. It is clear that the set $T(n)$ is increasing and $\bigcup_{n=0}^{\infty} T(n) = T$. So it is sufficient to prove $u_n(t) = \sum_{p \in T(n)} G^p(t) f^p$. If we notice that $T(n+1) = T(0) \cup \bigcup_{m=1}^{\infty} \{p = (p_1, \cdots, p_m)\,;\, p_1, \cdots, p_m \in T(n)\}$, then the equality is proved inductively by using the following formula:

$$\begin{aligned} &\sum_{p \in T(n+1)} G^p(t) f^p \\ &= e^{-t}f + \sum_{m=1}^{\infty} \sum_{p_1, \cdots, p_m \in T(n)} a_m \int_0^t e^{-(t-s)} G^{p_1}(s) \cdots G^{p_m}(s) B_m[f^{p_1}, \cdots, f^{p_m}] ds \\ &= e^{-t}f + \int_0^t e^{-(t-s)} B[\sum_{p \in T(n)} G^p(t) f^p] ds. \end{aligned}$$

Lemma 3. *$v(t) = u(t, f)(S)$ is the minimal solution of an equation*:

$$(10) \qquad \frac{dv(t)}{dt} = \sum_{n=1}^{\infty} a_n v(t)^n - v(t), \quad v(0+) = f(S), \quad 0 \leqq v(t) \leqq 1.$$

Proof. By the definition of $B[u]$, we have $B[u](S) = \sum_{n=1}^{\infty} a_n u(S)^n$, so from (8) $v(t)$ satisfies

$$(11) \qquad v(t) = e^{-t} f(S) + \int_0^t e^{-(t-s)} \sum_{n=1}^{\infty} a_n v(s)^n ds,$$

which is equivalent to (10). It is easily seen that $u_n(t)(S)$ is a sequence of successive approximation of (11) and $\lim_{n \to \infty} u_n(t)(S) = v(t)$, so $v(t)$ is the minimal solution of (10).

Proof of Theorem 1. It is clear that $u(t, f)$ satisfies (8), so also (1). Let $u(t)$ be any solution of (1). By (9) and Lemma 1, we have inductively $u_n(t) \leqq u(t)$, so $u(t, f) \leqq u(t)$ by Lemma 2, proving (a). It is well known that (10) has unique solution if and only if $f(S) < 1$, or $f(S) = 1$ and the condition (3) is satisfied [2]. So, b) and c) is proved by Lemma 3 if we notice that $u \geqq v$ with $u(S) = v(S)$ implies $u = v$. By Schauder-Tychonov theorem [1], there is $f_0 \in \mathfrak{W}_p$ such that $f_0 = B[f_0]$. By this initial condition, there is a trivial solution $u(t) \equiv f_0$ of (1).

If (3) is not satisfied, $v(t)$ is not trivial, that means $u(t) \neq u(t, f)$.

q.e.d.

4. Let $\hat{T} = T \cup \{\Delta\}$, where Δ is an extra point. Define the number $n(p)$ of surmits of the tree $p \in T$ by $n((\alpha)) = 1$ and $n(p) = \sum_{i=1}^{m} n(p_i)$ if $p = (p_1, \cdots, p_m)$. For each $p \in T$ such that $n(p) = n$ and $q_1, \cdots, q_n \in T$, $p(q_1, \cdots, q_n)$ is defined to be a tree given by replacing i-th (α) in p with q_i for each $i = 1, \cdots, n$. If for some i $(1 \leqq i \leqq n)$, $q_i = (\overbrace{(\alpha), \cdots, (\alpha)}^{j})$ with $j \geqq 1$ and $q_k = (\alpha)$ for $k \neq i$, then we write $p(i, j)$ instead of $p(q_1, \cdots, q_n)$.

Let $T_p = \{p(q_1, \cdots, q_n)\,;\, q_1, \cdots, q_n \in T\}$ and $\hat{T}_p = T_p \cup \{\Delta\}$. $C_0(\hat{T})$ (resp. $C_0(\hat{T}_p)$) is the set of all continuous functions on $\hat{T}$ (resp. $\hat{T}_p$) vanishing at Δ, where the topology of $\hat{T}$ is that of one-point compactification of the discrete topological space T. Let $X = \{X_t, P_p\,;\, p \in \hat{T}\}$ be a minimal Markov chain on $\hat{T}$ with Δ as a trap, having the generator:

(12) $\mathfrak{G}F(p) = \sum_{\substack{1 \leqq i \leqq n(p) \\ 1 \leqq j}} a_j F(p(i, j)) - n(p)F(p), \quad \text{for} \quad F \in C_0(T),\ p \in T.$

We may assume that X is a Hunt process with a Feller semi-group. Further, if we transform the state space of X by $n(p): \hat{T} \to \{1, 2, \cdots \cdots, \infty\}$, then X becomes the Galton-Watson process of continuous time parameter with generator:

$$\tilde{\mathfrak{G}}F(n) = n \sum_{j \geqq 1} a_j F(n + j - 1) - nF(n), \quad \text{for} \quad F \in C_0(\{0, 1, \cdots, \infty\}).$$

So, $P_p(e_\Delta = +\infty) = 1$ if and only if (3) is satisfied, where $e_\Delta = \inf\{t : X_t = \Delta\}$ [2]. But in the following, we do not assume (3).

Theorem 2. *$u(t, f) = E_{(\alpha)}(f^{X_t})$, where $f^\Delta = 0$.*

Proof. Assume that the following Lemma 4 is proved. Then, for $p = (p_1, \cdots, p_m) \in T$, and for first jumping time τ_1,

$$P_{(\alpha)}(X_t = p) = E_{(\alpha)}(P_{X_{\tau_1}}(X_{t-s} = p)|_{s=\tau_1}\,;\, X_{\tau_1} = (\overbrace{(\alpha), \cdots, (\alpha)}^{m}),\ \tau_1 \leqq t)$$
$$= a_m \int_0^t e^{-(t-s)} \prod_{k=1}^{m} P_{(\alpha)}(X_s = p_k) ds,$$

so, we can prove inductively that $G^p(t) = P_{(\alpha)}(X_t = p)$, that is, $u(t, f) = E_{(\alpha)}(f^{X_t})$.

Lemma 4. *For $p \in T$ such that $n(p) = n$ and $q_1, \cdots, q_n \in T$,*

(13) $$P_p(X_t = p(q_1, \cdots, q_n)) = \prod_{k=1}^{n} P_{(\alpha)}(X_t = q_k).$$

Proof. Property (13) is an analogy to the branching property, so the proof is essentially the same as in [3]. For each $p \in T$ such that $n(p) = n$ and $f_1, \cdots, f_n \in C_0(\hat{T})$, we define $F(f_1, \cdots, f_n) \in C_0(\hat{T}_p)$ by $F(f_1, \cdots, f_n)(q) = \prod_{k=1}^{n} f_k(q_k)$ if $q = p(q_1, \cdots, q_n) \in T_p$. For each $p \in T$, $q \in T_p$, $r \geqq 0$, and $f \in C_0(T_p)$, we define $T_t^{(r)} f(q) = E_q(f(X_t)\,;\, \tau_r \leqq t < \tau_{r+1})$, where τ_r is the r-th jumping time of X_t. Then, it is sufficient to prove the following equality:

$$(14)\qquad T_t^{(r)}F(f_1,\cdots,f_n)(q)=\sum_{\substack{r_1,\cdots,r_n\geqq 0\\ r_1+\cdots+r_n=r}} F(T_t^{(r_1)}f_1,\cdots,T_t^{(r_n)}f_n)(q).$$

Before to prove (14), we give the following two lemmas.

Lemma 5. *Let* $\psi(q\,;ds,l)=P_q(\tau_1\in ds, X_{\tau_1}=l)$. *Then for each* $p\in T$ *such that* $n(p)=n$, $q=p(q_1,\cdots,q_n)\in T_p$ *and* $l=p(l_1,\cdots,l_n)\in T_q\subset T_p$,

$$(15)\qquad \psi(q\,;ds,l)=\sum_{k=1}^{n}\psi(q_k\,;ds,l_k)\prod_{\substack{h\neq k\\ 1\leqq h\leqq n}} p_{q_h}(\tau_1>s)\delta_{\{q_h\}}(l_h).$$

Proof. If $n(q)=m$ and $l=q(l_1,\cdots,l_m)$, then

$$(16)\qquad \psi(q\,;ds,l)=me^{-ms}ds\sum_{i=1}^{m}\sum_{j=1}^{\infty}\frac{1}{m}a_j\delta_{\{q(i,j)\}}(l)$$

$$=\sum_{i=1}^{m}\psi((\alpha)\,;ds,l^i)\prod_{\substack{j\neq i\\ 1\leqq j\leqq m}} P_{(\alpha)}(\tau_1>s)\delta_{\{(\alpha)\}}(l^j).$$

But it is clear that $l_k=q_k(l^{M_k+1},l^{M_k+2},\cdots,l^{M_{k+1}})$ where $M_k=\sum_{i=1}^{k-1}n(q_i)$, so, by applying (16) inversely,

$$=\sum_{k=1}^{n}\psi(q_k\,;ds,l_k)\prod_{\substack{h\neq k\\ 1\leqq h\leqq n}} P_{q_h}(\tau_1>s)\delta_{\{q_h\}}(l_h).$$

Lemma 6. *Let* $g^{(r)}(s)=T_s^{(0)}T_{t-s}^{(r)}f(q)$. *Then, for* $r\geqq 1$,

$$(17)\qquad g^{(r)}(s)=\int_s^t\sum_{l\in T_q}T_{t-\theta}^{(r-1)}f(l)\psi(q\,;d\theta,l).$$

Proof. By the strong Markov property,

$$\begin{aligned} g^{(r)}(s)&=E_q(f(X_t)\,;\tau_r\leqq t<\tau_{r+1},s<\tau_1)\\ &=E_q(E_{X_{\tau_1}}(f(X_{t-\theta})\,;\tau_{r-1}\leqq t-\theta<\tau_r)|_{\theta=\tau_1}\,;s<\tau_1\leqq t)\\ &=\int_s^t\sum_{l\in T_q}T_{t-s}^{(r-1)}f(l)\psi(q\,;ds,l). \end{aligned}$$

Proof of (14). We prove (14) by induction for r. The case of $r=0$ is clear if we notice that $T_t^{(0)}f(q)=f(q)P_q(\tau_1>t)$. Let (14) be proved in the case of r. Then, by Lemmas 4 and 5,

$$\begin{aligned} T_t^{(r+1)}F(f_1,\cdots,f_n)(q)&=\int_0^t\sum_{l\in T_q}T_{t-s}^{(r)}F(f_1,\cdots,f_n)(l)\psi(q\,;ds,l)\\ &=\sum_{\substack{r_1,\cdots,r_n\geqq 0\\ r_1+\cdots+r_n=r}}\sum_{k=1}^{n}\int_0^t\sum_{l_k\in T_{q_k}}T_{t-s}^{(r_k)}f_k(l_k)\psi(q_k\,;ds,l_k)\prod_{\substack{h\neq k\\ 1\leqq h\leqq n}}T_s^{(0)}T_{t-s}^{(r_h)}f_n(q_h)\\ &=\sum_{\substack{r_1,\cdots,r_n\geqq 0\\ r_1+\cdots+r_n=r+1}}\prod_{k=1}^{n}g_k^{r_k}(0) \end{aligned}$$

so, the case of $r+1$ is proved.

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