199. *On a Problem of MacLane*

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1. Let $f(z)$ be a non-constant holomorphic function in $\{|z|<1\}$, having asymptotic values at each point of a dense subset on $\{|z|=1\}$. Such a function is said to belong to the class $\mathcal{A}$ (MacLane [1]). MacLane proposed a problem:

If $f(z)$ and $g(z)$ belong to $\mathcal{A}$, do $f(z)+g(z)$ and $f(z)g(z)$ belong to $\mathcal{A}$?

Ryan and Barth [2] answered to this negatively, and raised a further question:

If $f(z)\in\mathcal{A}$ and $b(z)$ is bounded, are $b(z)f(z)\in\mathcal{A}$? (We suppose, of course, that $b(z)f(z)$ is not a constant.)

In the present note, we will answer to this positively but only partly. That is, we will prove the following

Theorem A. *Let $b(z)$ be a function, holomorphic and bounded in $\{|z|<1\}$, having non-zero Fatou limits on $\{|z|=1\}$ except on a set of the first Baire category. Then, if $f(z)\in\mathcal{A}$, we have $b(z)f(z)\in\mathcal{A}$.*

2. For the sake of convenience, we repeat the definitions due to MacLane [1], with slight modifications in notations.

An arc $\Gamma: z=z(t)$, $0\leqq t<1$, in $\{|z|<1\}$ is said to be the path ending at a point ζ, $|\zeta|=1$, if $z(t)\to\zeta$ as $t\to 1$. A function $f(z)$ is said to have an asymptotic value a ($a=\infty$ permitted) at ζ, if there exists a path Γ ending at ζ on which $f(z)$ has the limit a, i.e., if $f(z(t))\to a$ as $t\to 1$. The set of these points is denoted by $A_f(a)$. That is, $A_f(a)$ is the set at each point of which $f(z)$ has the asymptotic value a. We put

$$A_f^*=\bigcup_{a\neq\infty}A_f(a),\qquad A_f=A_f^*\cup A_f(\infty).$$

A function $f(z)$ is defined to belong to the class $\mathcal{A}$ if $f(z)$ is holomorphic and non-constant in $\{|z|<1\}$ and the set A_f is dense on $\{|z|=1\}$.

Next we define the sets B_f^* and B_f. A point ζ, $|\zeta|=1$, belongs to B_f^* if and only if there exists a path Γ ending at ζ, on which $f(z)$ is bounded by some finite constant M. The bound M may vary as ζ and Γ vary. We put

$$B_f=B_f^*\cup A_f(\infty).$$

$f(z)$ is defined to belong to the class $\mathcal{B}$ if $f(z)$ is holomorphic and non-constant in $\{|z|<1\}$ and the set B_f is dense on $\{|z|=1\}$.

The set $\{z\,;\,|f(z)|=\lambda\}$, where $\lambda\geqq 0$ is a constant, is called *level set* and denoted by $L_f(\lambda)$. For each r, $0<r<1$, let the components of

$$L_f(\lambda)\cap\{r<|z|<1\}$$

be $\Lambda_i(r)$, $i\in I$. Let $\delta_i(r)=$diam. of $\Lambda_i(r)$ and put

$$\delta(r)=\sup_{i\in I}\delta_i(r)$$

with $\delta(r)\equiv 0$ if I is void. Clearly $\delta(r)\searrow$ as $r\nearrow$. We shall say that the level set $L_f(\lambda)$ ends at points of $\{|z|=1\}$ if and only if $\delta(r)\searrow 0$ as $r\nearrow 1$.

$f(z)$ is defined to belong to the class $\mathcal{L}$ if $f(z)$ is holomorphic and non-constant in $\{|z|<1\}$ and every level set $L_f(\lambda)$ ends at points of $\{|z|=1\}$.

MacLane proved the following important

Theorem M. $\mathcal{A}=\mathcal{B}=\mathcal{L}$.

3. Now we prove our Theorem A. Suppose that $b(z)f(z)\notin\mathcal{A}$. By Theorem M, $b(z)f(z)\notin\mathcal{B}$ and hence there exists an arc γ on $\{|z|=1\}$ such that

(3.1) $$B_{bf}\cap\gamma=\phi.$$

Since a fortiori

(3.2) $$B^*_{bf}\cap\gamma=\phi,$$

$B^*_f\cap\gamma$ must be void. Then there exists a sequence of arcs $\{C_n\}$ in $\{|z|<1\}$ such that (see [1], p. 15).

(3.3) $C_n\cap C_m=\phi$ if $n\neq m$; $C_n\to\gamma$ and $\inf_{z\in C_n}|f(z)|\to\infty$ as $n\to\infty$.

Let

$$\mu_n=\inf_{z\in C_n}|f(z)|,$$
$$\gamma=\{e^{i\theta};\ \alpha\leqq\theta\leqq\beta\},$$
$$S=\{z;\ |z|<1,\ \alpha<\arg z<\beta\}.$$

By choosing γ suitably, we may assume that

(3.4) C_n is a cross-cut of the sector S and, if $n>m$, C_n separates C_m from γ.

Then ([1], Theorem 3) $\gamma\subset A_f(\infty)$, i.e., for any point $\zeta\in\gamma$ there is a path $\Gamma(\zeta)$ ending at ζ such that $f(z)\to\infty$ on $\Gamma(\zeta)$. But because of (3.1)

$$\varliminf_{\Gamma(\zeta)}|b(z)f(z)|<+\infty.$$

Take α', β' $(\alpha<\alpha'<\beta'<\beta)$ and put

$$\gamma'=\{e^{i\theta};\ \alpha'\leqq\theta\leqq\beta'\}.$$

For a natural number N we set

(3.5) $E_N=\{\zeta\in\gamma'$; there exists a path $\Gamma(\zeta)$ ending at ζ, on which $f(z)\to\infty$ and $\varliminf|b(z)f(z)|\leqq N\}$.

E_N is a closed set. To prove this, let $\zeta_n\in E_N$ and $\zeta_n\to\zeta_0$. We will construct a path $\Gamma(\zeta_0)$ satisfying the condition (3.5).

For each n, we can easily find a point $z_n\in\Gamma(\zeta_n)$ such that

(3.6) $$|z_n-\zeta_n|<\frac{1}{n},\quad |f(z_n)|\geqq\mu_n,\quad |b(z_n)f(z_n)|<N+\frac{1}{n}.$$

Then

(3.7) $z_n \to \zeta_0$ and $\underline{\lim}\,|b(z_n)f(z_n)| \leqq N,$ as $n \to \infty.$

We may assume that $\alpha < \arg z_n < \beta$ and $|z_n| < |z_{n+1}|$, $n=1, 2, \cdots$. Connecting these points by segments in order, we get a path (Jordan arc) Γ' which tends monotonely to $\{|z|=1\}$, lying in the sector $\alpha < \arg z < \beta$, and ends at ζ_0.

Let $\alpha_k = \arg \zeta_0 - \dfrac{1}{k}$, $\beta_k = \arg \zeta_0 + \dfrac{1}{k}$, and let $R(\alpha_k)$, $R(\beta_k)$ be the radii to $e^{i\alpha_k}$, $e^{i\beta_k}$ respectively.

Let $E(n, k)$ be the domain bounded by C_n, C_{n+1}, $R(\alpha_k)$, $R(\beta_k)$, and let the components of $L_f(\mu_k) \cap E(n, k)$ be $l_f(n, k; i)$. If $n > k$, each of $l_f(n, k; i)$ is apart from C_n and C_{n+1}. Put

(3.8) $\delta_{n,k} = \text{max. diam. of } 1_f(n, k; i).$

Since $f(z) \in \mathcal{L}$

(3.9) $\delta_{n,k} \to 0$ as $n \to \infty$ for any fixed k.

Let $k=1$. There exists an n_1 such that if $n \geqq n_1$, any curve $l_f(n, k; i)$ which intersects with Γ' is contained in $E(n, 1)$. Hence any portions of Γ' in $E(n, 1)$, on which $|f(z)| < \mu_1$, may be replaced by Jordan subarcs of $l_f(n, 1; i)$. Making such replacements (finite in number for any n) for each $n \geqq n_1$, we obtain a path Γ_1 such that

$$\underline{\lim}\,|f(z)| \geqq \mu_1 \quad \text{on} \quad \Gamma_1.$$

Γ_1 tends to ζ_0 and contains all z_n, so that on Γ_1 $\underline{\lim}\,|b(z)f(z)| \leqq N$.

Next we find an n_2 such that if $n > n_2$, any curve $l_f(n, 2; j)$ which intersects with Γ_1 is contained in $E(n, 2)$. Hence any portions of Γ_1 in $E(n, 2)$, on which $|f(z)| < \mu_2$, may be replaced by Jordan subarcs of $l_f(n, 2; j)$ and we obtain a path Γ_2 which tends to ζ_0 and contains all z_n. Similarly, we can construct $\Gamma_3, \Gamma_4, \cdots$.

Continuing this procedure indefinitely, we obtain a path $\Gamma(\zeta_0)$ which obviously has the required property (3.5).

4. Because of (3.1), we have $\bigcup_N E_N = \gamma'$. Since E_N, $N=1, 2, \cdots$ are closed, some E_N, say E_{N_1}, must contain an arc γ^* by the theorem of Baire. For every $\zeta \in \gamma^*$ there is a sequence $z_n = z_n(\zeta)$, $n=1, 2, \cdots$ such that $z_n \to \zeta$ and

(4.1) $|f(z_n)| \geqq \mu_n, \qquad |b(z_n)f(z_n)| \leqq N_1 + \dfrac{1}{n}.$

Let $\{\zeta_l\}$ be a countable set, dense on γ^*. Write $z_n(\zeta_l) = z_{n,l}$. Then, from (4.1) we have

(4.2) $|b(z_{n,l})| \leqq \dfrac{2N_1}{\mu_n},$ whatever l may vary.

$$\begin{array}{ccccccccc} z_{1,1} \rightarrow & z_{2,1} & \nearrow z_{3,1} & & \nearrow z_{4,1} & \cdots\cdots & z_{n,1} & \cdots \rightarrow & \zeta_1, \\ & \downarrow & \swarrow \downarrow & & \downarrow & & & & \\ z_{1,2} & z_{2,2} & z_{3,2} & \swarrow & z_{4,2} & \cdots\cdots & z_{n,2} & \cdots \rightarrow & \zeta_2, \\ & & \downarrow & & \downarrow & & & & \\ z_{1,3} & z_{2,3} & z_{3,3} & & z_{4,3} & \cdots\cdots & z_{n,3} & \cdots \rightarrow & \zeta_3, \\ & & & & \downarrow & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \rightarrow & . \end{array}$$

From the double sequence $\{z_{n,l}, n \geqq l\}$ we form a sequence $\{Z_n\}$ as shown in the above figure, i.e.,

$$Z_1 = z_{1,1}, \quad Z_2 = z_{2,1}, \quad Z_3 = z_{2,2}, \quad Z_4 = z_{3,1}, \quad Z_5 = z_{3,2}, \cdots.$$

By (4.2), $\{Z_n\}$ has the following properties:

(4.3) For any subsequence $\{Z_{n_k}\}$ of $\{Z_n\}$, $b(Z_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$;

(4.4) For any point $\zeta \in \gamma^*$ and any $\varepsilon > 0$, there is a Z_n such that

$$|\zeta - Z_n| < \varepsilon.$$

5. Let $V(\varphi, \zeta)$ be a Stolz domain with vertex at $\zeta = e^{i\theta}$ and with opening 2φ:

$$V(\varphi, \zeta) = \{z ; |z| < 1, |\arg(1 - ze^{-i\theta})| < \varphi\}.$$

We will show that the set

$F = \{\zeta \in \gamma^* ;$ for any φ, $V(\varphi, \zeta)$ contains only finitely many points of Z_n's$\}$ is of the first Baire category on γ^*.

Let K be an integer and put

$F(\varphi, K) = \{\zeta \in \gamma^* ; V(\varphi, \zeta)$ contains exactly K points of Z_n's$\}$.

Since $F = \bigcap_{0<\varphi<\frac{\pi}{2}} \{\bigcup_{K \geqq 0} F(\varphi, K)\}$, it suffices to show that $F(\varphi, K)$ is nowhere dense on γ^* for fixed φ and K.

Take a subarc $\hat{\gamma} \subset \gamma^*$. If $\hat{\gamma} \cap F(\varphi, K) \ni \hat{\zeta} = e^{i\psi}$, $V(\varphi, \hat{\zeta})$ contains Z_{n_i}, $i = 1, 2, \cdots, K$. Let $L_1 = \{z ; \arg(1 - ze^{-i\psi}) = -\varphi\}$ and $L_2 = \{z ; \arg(1 - ze^{-i\psi}) = \varphi\}$ be the sides of $V(\varphi, \hat{\zeta})$, and let Z_{n_1}, Z_{n_2} be the points nearest to L_1, L_2 respectively. Let $\hat{\zeta}_1 = e^{i\varphi_1}$ and $\hat{\zeta}_2 = e^{i\varphi_2}$ be the points such that $\arg(1 - Z_{n_1}e^{-i\varphi_1}) = -\varphi$, $\arg(1 - Z_{n_2}e^{-i\varphi_2}) = \varphi$. By (4.4) there is a point Z_m such that $\varphi_1 < \arg Z_m < \varphi_2$ and $Z_m \notin V(\varphi, \hat{\zeta})$. Let $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = e^{i\theta_2}$ be the points such that $\arg(1 - Z_m e^{-i\theta_1}) = -\varphi$, $\arg(1 - Z_m e^{-i\theta_2}) = \varphi$. If $|Z_m|$ is sufficiently near to 1, the arc $\hat{\gamma}_1 = \{e^{i\theta} ; \theta_1 \leqq \theta \leqq \theta_2\}$ is contained in the arc $\{e^{i\theta} ; \varphi_1 \leqq \theta \leqq \varphi_2\}$. Hence for any $\zeta \in \hat{\gamma}_1$, $V(\varphi, \zeta)$ contains $(K+1)$ points $Z_{n_1}, Z_{n_2}, \cdots, Z_{n_K}$ and Z_m, so that $\zeta \notin F(\varphi, K)$ and $\hat{\gamma}_1 \cap F(\varphi, K) = \phi$. This shows that $F(\varphi, K)$ is nowhere dense and F is of the first category.

Hence the set $H = \gamma^* \setminus F$ is of the second category. If $\zeta \in H$, $V(\varphi, \zeta)$ contains infinitely many points of Z_n's for some φ. Thus if we put

$$H_1 = \{\zeta \in H ; b(z) \text{ has the Fatou limit 0 at } \zeta\},$$

$$H_2 = \{\zeta \in H ; b(z) \text{ has no Fatou limit at } \zeta\},$$

then $H = H_1 \cup H_2$. But by our assumption, H_1 and H_2 must be of the first category. This contradiction proves our theorem.

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References

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[2] Ryan, F. B., and K. F. Barth: Asymptotic values of functions holomorphic in the unit disk. Math. Z., **100**, Heft 5, 414–415 (1967).