198. Oscillatory Property of Certain Non-linear Ordinary Differential Equations. II

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1. The author [2] recently extended certain theorems of Kartsatos [1] and proved that certain even-order differential equations of the form

$$x^{(2n)}+f(t)g(x, x', \cdots, x^{(2n-1)})=0 \tag{1}$$

can have only oscillatory solutions. Non-oscillatory solutions may occur, however, for odd-order differential equations of the same form

$$x^{(2n+1)}+f(t)g(x, x', \cdots, x^{(2n)})=0 \tag{2}$$

when f and g satisfy the same conditions as in the even-order case, and so it becomes a problem how to distinguish oscillatory solutions of the equations (2). (This was pointed out in a discussion with Professors Nakashima and Sugiyama.)

It is indeed possible to prove oscillation theorems even for the equations of the form (2). Roughly speaking, our oscillation theorems assert that any solution of an equation of the form (2) with the same conditions on f and g as in [2] is oscillatory whenever it has at most one zero and n is any positive integer.

Similar oscillation theorems have recently been obtained by Waltman [3] for third order equations under different assumptions.

2. We shall consider the differential equation (2) and the following conditions:

(α) *f is positive function defined on the interval $\bar{I}=[t_0, +\infty)$ with $t_0\geqslant 0$ and $\int_{t_0}^{+\infty}f(t)dt=+\infty$;*

(β) *g is defined on R^{2n+1}; $sgn\ g(x_1, x_2, \cdots, x_{2n+1})=sgn\ x_1$ for any $(x_1, x_2, \cdots, x_{2n+1})\in R^{2n+1}$; and $g(\lambda x_1, \lambda x_2, \cdots, \lambda x_{2n+1})=\lambda^{2p+1}g(x_1, x_2, \cdots, x_{2n+1})$ for any $(x_1, x_2, \cdots, x_{2n+1})\in R^{2n+1}$, and $\lambda\in R$ and some nonnegative integer p;*

(γ) *g is defined on R^{2n+1}; $sgn\ g(x_1, x_2, \cdots, x_{2n+1})=sgn\ x_1$ for any $(x_1, x_2, \cdots, x_{2n+1})\in R^{2n+1}$; $g(-x_1, -x_2, \cdots, -x_{2n+1})=-g(x_1, x_2, \cdots, x_{2n+1})$ for any $(x_1, x_2, \cdots, x_{2n+1})\in R^{2n+1}$; and for any $2\leqslant k\leqslant 2n$ and $c\geqslant 0$, the function $g(x_1, x_2, \cdots, x_{2n+1})$ has a definit limit $G(k, c)$, which is positive or $+\infty$, as $x_1\to+\infty, \cdots, x_{k-1}\to+\infty, x_k\to c, x_{k+1}\to 0, \cdots, x_{2n+1}\to 0$;*

where all functions considered are real-valued and continuous on their domains. Then our theorems read as follows.

Theorem 1. *Suppose that the equation* (2) *satisfies* (α) *and* (β), *where* $n \geqslant 1$. *Then, any solution of* (2) *on the interval* $\bar{I}$ *is oscillatory, whenever it has a zero.*

Theorem 2. *Suppose that the equation* (2) *satisfies* (α) *and* (γ), *where* $n \geqslant 1$. *Then, any solution of* (2) *on the interval* $\bar{I}$ *is oscillatory, whenever it has a zero.*

In [2], we proved a lemma which played an important role in the proof of the oscillation theorems for the equation (1). But, for the differential equations of the form (2), the original lemma is not useful and we shall modify it so as to apply to the present situation.

Once we get a new lemma, the proof of the present oscillation theorems is entirely similar to that of earlier ones. So we shall sketch just the proof of the lemma and omit other details.

Lemma. *Let* φ *be a continuous function defined on* $\bar{I}=[t_0, +\infty)$, $(2n+1)$*-times continuously differentiable on* $I=(t_0, +\infty)$ *for* $n \geqslant 1$. *If* $\varphi>0$, $\varphi^{(2n+1)}<0$ *on* I *and* $\varphi(t_0)=0$, *then*

$$\lim_{t \to +\infty} \frac{\varphi^{(k)}(t)}{\varphi(t)}=0 \quad for \quad 1 \leqslant k \leqslant 2n.$$

For each $0 \leqslant k \leqslant 2n$, $\varphi^{(k)}$ *has a definite limit as* $t \to +\infty$. *If we denote by* $\varphi^{(k)}(\infty)$ *these limits, then there happen only the following cases*: $\varphi(\infty)=\cdots=\varphi^{(k-1)}(\infty)=+\infty$, $\varphi^{(k)}(\infty)=c \geqslant 0$, $\varphi^{(k+1)}(\infty)=\cdots=\varphi^{(2n)}(\infty)=0$ *with* $k \geqslant 1$.

Proof of Lemma. As $\varphi^{(2n+1)}<0$ on I, $\varphi^{(2n)}$ is decreasing on I, so that $\varphi^{(2n)}$ has a limit, finite or $-\infty$, as $t \to +\infty$. As was shown in [2], if $\varphi^{(2n)}(\infty)<0$, we get a contradiction and if $\varphi^{(2n)}(\infty)>0$, then we get the lemma. If $\varphi^{(2n)}(\infty)=0$, then $\varphi^{(2n)}>0$ on I and therefore $\varphi^{(2n-1)}$ is increasing on I.

If $\varphi^{(2n-1)}(\infty)<0$, we get a contradiction and if $\varphi^{(2n-1)}(\infty)>0$, the lemma is true. If $\varphi^{(2n-1)}(\infty)=0$, then $\varphi^{(2n-1)}<0$ on I and $\varphi^{(2n-2)}$ is decreasing on I. Here we may also have three cases:

$$\varphi^{(2n-2)}(\infty)>0, \ \varphi^{(2n-2)}(\infty)<0 \ \text{or} \ \varphi^{(2n-2)}(\infty)=0.$$

In the first case the lemma is true. The second case never happens. So we have only to dispose of the third case.

Repeating similar arguments, we shall have only the following case that remains to be unsettled:

$$(-1)^k \varphi^{(k)}>0 \ \text{and} \ \varphi^{(k)}(\infty)=0 \ \text{for} \ 1 \leqslant k \leqslant 2n.$$

In this case, $\varphi'<0$ on I and so φ is decreasing on I. As $\varphi(t_0)=0$ by hypothesis, $\varphi(t)<0$ on I, which contradicts the hypothesis that $\varphi(t)>0$ on I.

Thus we have $\varphi^{(k)}(\infty)>0$ for some $k \geqslant 1$. Hence our lemma is completely proved.

References

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[2] H. Onose: Oscillatory property of certain non-linear ordinary differential equations. Proc. Japan Acad., **44**, 110–113 (1968).
[3] P. Waltman: Oscillation criteria for third order non-linear differential equations. Pacific J. Math., **18**, 385–389 (1966).