197. *Inertia Groups of Low Dimensional Complex Projective Spaces and Some Free Differentiable Actions on Spheres. I*

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1. Introduction and preliminary lemmas. Sullivan has proved that the concordance classes of smoothing the combinatorial complex projective space is in one-to-one correspondence with the c-orientation preserving diffeomorphism classes where c is the generator of $H^2(CP^n)$ (see [6]). The conjugation map $g:(e_0, \cdots, e_n)\to(\bar{e}_0, \cdots, \bar{e}_n)$ (the complex conjugation) induces the diffeomorphism $g: CP^n\to CP^n$ such that $g_*(c) = -c$. Let $s:[CP^n, PD/O]\to\mathcal{S}(CP^n)$ be the natural correspondence from the concordance classes to the smooth structures. If $s(c_1)=CP^n$ (the natural smooth structure) and $s(c_2)=CP'^n$ and if there exists a diffeomorphism $d: CP^n\to CP'^n$ such that $d_*(c)=-c$ (where c is determined by the concordance class), then $(dg)_*(c)=d_*g_*(c)=d_*(-c)=c$, i.e., the composed diffeomorphism $d\cdot g$ induces the c-orientation preserving diffeomorphism. This implies that two concordance classes c_1, c_2 such that $s(c_1)=s(c_2)=CP^n$ are equivalent.

The inertia group of a smooth manifold M^n is interpreted as follows. (For the definition of the inertia group, see [5]). We may assume that the smooth structure M^n corresponds to the zero element $0\in[M, PD/O]$.

Lemma 1. $$I(M^n)=(sj)^{-1}(M^n)$$ *where j denotes the homomorphism of the Puppe's exact sequence*

$$\to[M/M\text{-Int}\, D, PD/O]\xrightarrow{j}[M, PD/O]\to[M\text{-Int}\, D, PD/O]\to.$$

Therefore, to study the inertia group $I(CP^n)$, we have only to study the following Puppe's exact sequence,

$$\to[SCP^{n-1}, PD/O]\xrightarrow{\partial}[S^{2n}, PD/O]\xrightarrow{j}[CP^n, PD/O]\to.$$

Let f be the attaching map $f:\partial e^{2n}\to CP^{n-1}$ of the $2n$-cell e^{2n} in CP^n and $S(f)$ be its suspension map. Then we shall have

Lemma 2. $$\partial=\{S(f)\}^*$$ *where $\{S(f)\}^*$ denotes the homomorphism induced by $S(f)$.*

It is well-known that every free differentiable action of S^1 (or S^3) on a homotopy sphere $\tilde{S}^n$ is always a principal fibration (see [2]) and that this fibration is homotopically equivalent to the classical Hopf fibration (see [4]). Therefore the bundle-theoretic approach to smoothing problem of Hirsch and Mazur (see [3]) enables us to study the differentiable free actions.

Detailed proof will appear elsewhere.

2. Statement of results. Using the fibration: $PD/O \to F/O \to F/PD$, we shall have

Theorem 1. *The inertia group of the complex projective space, $I(CP^n)$, is trivial for $n \leqq 8$.*

Remark. Sullivan has proved that $I(CP^4)=0$ (see [1]). In case $n=8$, this is suggested to the author by Professor H. Toda.

Any differentiable free S^1-action on a homotopy sphere $\tilde{S}^{2n+1}$ is a principal fibration: $S^1 \to \tilde{S}^{2n+1} \xrightarrow{p} \tilde{S}^{2n+1}/\varphi$ and we consider the associated disk bundle: $D^2 \to \tilde{B}^{2n+2} \to \tilde{S}^{2n+1}/\varphi$. Since the boundary $\partial \tilde{B}^{2n+2}$ of the total space is PL-homeomorphic to the sphere, we can construct a PL-manifold $\tilde{B}^{2n+2} \cup e^{2n+2} = X$. If the orbit space $\tilde{S}^{2n+1}/\varphi$ is PL-homeomorphic to the complex projective space CP^n, X is PL-homeomorphic to the complex projective space CP^{n+1}. Consequently we shall have

Theorem 2. *A homotopys phere $\tilde{S}^{2n+1}$ admits a differentiable free S^1-action such that the orbit space is PL-homeomorphic to CP^n if and only if $\tilde{S}^{2n+1}$ corresponds to a composition $S^{2n+1} \xrightarrow{f} CP^n \xrightarrow{g} PD/O$ for some map g, by the natural isomorphism $\Theta_{2n+1} \cong [S^{2n+1}, PD/O]$.*

As corollaries, we shall have

Corollary 1. *There exists no differentiable free action of S^1 on an exotic sphere $\tilde{S}^{2n+1} (\neq S^{2n+1})$ such that the orbit space is PL-homeomorphic to the complex projective space CP^n when n is any of* 3, 4, 8.

Corollary 2. $\tilde{S}^{13}$ *(of course, this does not bound a π-manifold) admits a differentiable free S^1-action such that the orbit space is PL-homeomorphic to the complex projective space CP^6.*

Corollary 3. *There exists an exotic* 15*-sphere $\tilde{S}^{15}$ which does not bound a π-manifold such that $\tilde{S}^{15}$ admits a free differentiable action of S^1.*

Let L^n be the n-dimensional PL-manifold with boundary ∂L such that ∂L is PL-homeomorphic to the sphere S^{n-1}. Let $K = L \cup e^n$ be the PL-manifold obtained by attaching a disk e^n. Then we shall have

Theorem 3. *If L has a smooth structure L_α such that ∂L_α does not correspond to the composition*

$$\partial L \subset L \xrightarrow{g} PD/O$$

for any map g by the natural isomorphism $\Theta_n \cong [S^n, PD/O]$, then $K = L \cup e$ has no smooth structure.

As an easy application, we shall have

Corollary. *There are infinitely many combinatorially distinct* 12*-manifolds which admit no smooth structure.*

Remark. These manifolds have the homotopy type of the complex projective space CP^6. And that this non smoothability follows from the different reason from that of Sullivan's examples (cf. Sullivan [6]).

References

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