195. On Concentric Semigroups*)

By Takayuki TAMURA
University of California, Davis, California, U.S.A.

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By a $\mathfrak{p}$-simple semigroup we mean a semigroup which has no prime ideal, equivalently an $\mathcal{S}$-indecomposable semigroup, i.e., a semigroup which has no semilattice-homomorphic image except a trivial one. (See [5].) A commutative semigroup S is $\mathfrak{p}$-simple if and only if it is commutative archimedean. (See [3], [4].) A commutative archimedean semigroup has one of the following types (cf. [6]):

(1) A commutative nil-semigroup (i.e., some power of every element is zero).

(2) An ideal extension of an abelian group G, $|G|>1$, by a semigroup Z, $|Z|\geqq 1$, of type (1).

(3) A commutative archimedean torsion-free semigroup (i.e., having no idempotent).

Let $C=\bigcap_{n=1}^{\infty} Sa^n$. If S is commutative, C is the kernel (i.e., the minimal ideal) and hence does not depend on a and

$$C=\{0\} \text{ for (1)}; \quad C=G \text{ for (2)}; \quad C=\phi \text{ for (3)}.$$

Let S be a commutative archimedean semigroup and define a relation $\preceq$ by divisibility, i.e.,

$$a\preceq b \text{ iff either } a=b \text{ or } a=bx \text{ for some } x\in S.$$

The relation is a partial ordering on S if and only if either $C=\{0\}$ or $C=\phi$.

Thus C plays an important role in commutative archimedean semigroups. This concept, however, can be defined in any semigroup though it depends on elements. It will be called the "closet" of an element a. In this note we will report some of the results of the study of semigroups with constant closet without proof. The proof will be published elsewhere [7].

Definition. Let S be a semigroup and let $a\in S$.

(4) $C_l(a)=\bigcap_{n=1}^{\infty} Sa^n$ is called the left closet of a in S.

(5) $C_r(a)=\bigcap_{n=1}^{\infty} a^nS$ is called the right closet of a in S.

(6) $C(a)=\bigcap_{n=1}^{\infty} Sa^nS$ is called the closet of a in S.

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These could be empty. S is left (right) concentric if $C_l(a)(C_r(a))$ is constant. S is concentric if $C(a)$ is constant. An element a of S is finitely convergent if $C(a) \neq \phi$ and $C(a) = Sa^mS$ for some m. S is finitely convergent if all elements of S are finitely convergent. An element a is divergent if $C(a) = \phi$. S is divergent if all elements are divergent. S is $*$-finitely convergent if each element of S is either finitely convergent or divergent.

Theorem 1. *A semigroup S is concentric and finitely convergent if and only if S is an ideal extension of a simple semigroup by a nil-semigroup.*

Accordingly, a concentric finitely convergent semigroup is $\mathfrak{p}$-simple, but the converse is not true.

Theorem 2. *Let $\mathfrak{S}$ be a class of semigroups. Suppose that $\mathfrak{S}$ satisfies*

(7) *If $S \in \mathfrak{S}$ every Rees-factor semigroup of S is*[1] *in $\mathfrak{S}$.*

Under (7) *the following two conditions on $\mathfrak{S}$ are equivalent.*

(8) *Every semigroup in $\mathfrak{S}$ is concentric and $*$-finitely convergent.*

(9) *If $S \in \mathfrak{S}$ and if S has a zero, S is a nil-semigroup.*

Furthermore, every S in $\mathfrak{S}$ is $\mathfrak{p}$-simple; an ideal of each S in $\mathfrak{S}$ is concentric and $*$-finitely convergent; homomorphic image of S in $\mathfrak{S}$ is concentric and $*$-finitely convergent.

Theorem 3. *A semigroup S is right concentric, $C_r(a) = T \neq \phi$ for all $a \in S$, if and only if*

(10) *T is an ideal of S and right simple and*

(11) *S/T is a right concentric semigroup with zero.*

Theorem 4. *A semigroup S is a concentric semigroup with closet $C(\neq \phi)$ if and only if S is an ideal extension of a simple semigroup C by a concentric semigroup Z with zero.*

Remark 1. Given $\mathfrak{S}$ mentioned in Theorem 2 there is a smallest class $\mathfrak{S}'$, containing $\mathfrak{S}$, which satisfies:

(12) Every semigroup S in $\mathfrak{S}'$ is concentric and $*$-finitely convergent.

(13) If $S \in \mathfrak{S}'$, a homomorphic image of S is in $\mathfrak{S}'$.

(14) If $S \in \mathfrak{S}'$, an ideal of S is in $\mathfrak{S}'$.

Remark 2. If I is a divergent semigroup and if Z is a concentric semigroup with zero, then an ideal extension of I by Z is not necessarily concentric.

Examples of $\mathfrak{S}$.

(15) The class of all commutative archimedean semigroups.

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1) This means that $\mathfrak{S}$ contains a semigroup isomorphic to Rees-factor semigroup.

(16) The class of all medial archimedean semigroups.

A semigroup S is called medial if S satisfies the identity $xyzu = xzyu$. A medial semigroup S is called archimedean if for every $a, b \in S$ there are $x, y, z, u \in S$ and positive integers m and n such that

$$a^m = xby \text{ and } b^n = zau.$$

A medial semigroup is $\mathfrak{p}$-simple if and only if it is archimedean. These results are due to Chrislock [1], [2]. A medial archimedean semigroup S is concentric: If S contains a zero, S is a nil-semigroup; if S contains a non-zero idempotent, for all $a \in S$, $C(a)$ is the kernel and it is the direct product of a rectangular band and an abelian group [1], [2]; if S has no idempotent, S is divergent (see [8]).

References

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[7] ——: The study of closets and free contents related to semilattice decomposition of semigroups (to be published in Proc. of Semigroup Symposium, Academic Press).

[8] ——: Notes on medial archimedean semigroups without idempotent (to be published).