9. *Local Knots of 2-Spheres in 4-Manifolds*

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Throughout this paper we will only be concerned from the combinatorial point of veiw. By $(fM^2 \subset M^4)$ we denote a pair of manifolds such that M^4 is a triangulated oriented 4-dimensional manifold and fM^2 is a properly embedded oriented 2-dimensional manifold as a subcomplex in M^4 and f is a piesewise linear embedding of M^2 in M^4.

We measure the *local knot type** of the embedding f at an interior point x of M^2 as follows, [1], [3]. Let $\mathrm{St}(fx, M^4)$ and $\mathrm{St}(x, M^2)$ denote the closed star neighborhoods of fx in M^4 and x in M^2 respectively. The boundary** $S^3 = \partial \mathrm{St}(fx, M^4)$ of $\mathrm{St}(fx, M^4)$ is a 3-sphere with an orientation inherited from that of M^4, and the boundary $S^1 = \partial \mathrm{St}(x, M^2)$ is a 1-sphere with an orientation inherited from that of M^2. The oriented knot type (denote $\kappa(x)$) of the embedding of fS^1 in S^3 is called the *local knot type* of the embedding f at x. When $\kappa(x)$ is of trivial type, we may say that the *local knot type is O* or that fM^2 is *locally flat* (*unknotted*) at fx. A 2-manifold fM^2 is called *locally flat* if it is locally flat at each of its points. When $\kappa(x)$ is of non-trivial type, we may say that fM^2 is *locally knotted* at fx or that fx is locally knotted point of fM^2.

Of course the local knot type can also be measured at a boundary point $x \in \partial M^2$. In this case $\mathrm{cl}(\partial \mathrm{St}(fx; M^4) \cap \mathcal{J} M^4)$ and $\mathrm{cl}(\partial \mathrm{St}(x, M^2) \cap \mathcal{J} M^2)$ are 3-cell and 1-cell respectively and the local knot type is a type of (1, 3)-cell pair. In this paper we shall consider only embeddings whose boundary points are all locally flat (unknotted).

Since a locally knotted point must be a vertex in any triangulation of the pair $(fM^2 \subset M^4)$ the locally knotted points are always isolated. If M^2 is compact, there can be only a finite number of locally knotted points.

R. H. Fox and J. W. Milnor observed "Under which condition can a given collection of knot types $\kappa_1, \cdots, \kappa_n$ be the set of local knot types of some embedding of a 2-sphere S^2 in the 4-space R^4?" and defined the slice knot types and showed that a collection $\kappa_1, \cdots, \kappa_n$ of knot types can occur as the collection of local knot types of a 2-sphere

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*) R. H. Fox and J. W. Milnor called it the local *singularity* [1], but as it is confused with the self-intersection (socalled singularity) we'll use this terminology.

**) ∂=boundary, $\mathcal{J}$=interior, cl=closure.

in 4-space if and only if $\kappa_1+\cdots+\kappa_n$ is the type of a slice knot.

By the unique decomposition theorem of the knot [5], this problem is reduced to the following: Which knot types can occur as the only local knot type of a 2-sphere fS^2 in R^4? The purpose of the paper is to study the following problem: *Which knot types κ can occur as the only local knot type of a 2-sphere fS^2 in some kinds of 4-manifold M^4?*

Let $K(fS^2\subset M^4)$ be the set of knot types which can occur as the only one local knot type of a pair $(fS^2\subset M^4)$. Then we now state our version of Fox-Milnor's theorem.

Theorem 0. *$K(fS^2\subset R^4)=K(fS^2\subset S^4)=K(fS^2\subset D^4)=\{$slice knot types$\}$, where R^4, S^4, and D^4 are 4-space, 4-sphere and 4-cell respectively.*

Moreover, since $S^1\times D^3$, $S^2\times D^2$, and $S^1\times S^1\times D^2$ are realizable in R^4(or S^4, D^4) we have:

Theorem 0′. $K(fS^2\subset S^1\times D^3)=K(fS^2\subset S^2\times D^2)$
$=K(fS^2\subset S^1\times S^1\times D^2)$
$=\{$*slice knot types*$\}$.

The following will be established in § 2.

Theorem 1. $K(fS^2(S^2\times S^2)=\{$*all knot types*$\}$.

§ 1. Application of Fox-Milnor's Theorem. We apply the Fox-Milnor's Theorem to study $K(fS^2\subset S^1\times S^1\times S^1\times S^1)$ and $K(fS^2\subset S^1\times S^1\times S^2)$, where $S^1\times S^1\times S^1\times S^1$ and $S^1\times S^1\times S^2$ are not realizable in R^4. The universal covering spaces of $S^1\times S^1\times S^1\times S^1$ and $S^1\times S^1\times S^2$ are $R^1\times R^1\times R^1\times R^1$ and $R^1\times R^1\times S^2$ respectively. While $R^1\times R^1\times R^1\times R^1$ and $R^1\times R^1\times S^2$ are homeomorphic to R^4 and $\mathcal{J}(D^2\times S^2)$ respectively. Thus as an immediate consequence of Theorem 0 and Theorem 0′ we have:

Theorem 0″. $K(fS^2\subset S^1\times S^1\times S^1\times S^1)$
$=K(fS^2\subset R^1\times R^1\times S^2)$
$=K(fS^2\subset S^1\times S^1\times S^2)$
$=\{$*slice knot types*$\}$.

Moreover, since the universal covering space of any lens space L is the 3-sphere S^3, similarly we have:

Theorem 0‴. *For every lens space L,*
$K(fS^2\subset L\times R^1)=K(fS^2\subset L\times D^1)$
$=K(fS^2\subset L\times S^1)$
$=\{$*slice knot types*$\}$.

§ 2. Proof of Theorem 1. We shall need the following elementary lemma, due to H. Terasaka [6].

Lemma 1. *Let $u(\kappa)$ (≥ 1) be the unknotting number*** of a knot type k. Then there is a representative k of κ as follows. Let*

***) Überschneidungszahl, see K. Reidemeister [4], p. 17.

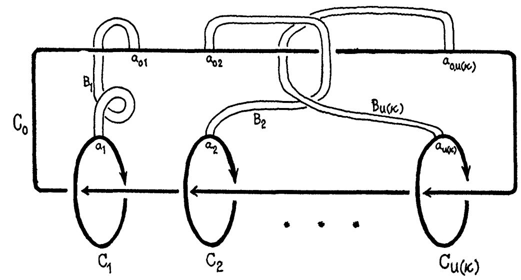

Fig. 1.

$c_0, c_1, \cdots, c_{u(\kappa)}$ be $u(\kappa)+1$ unknotted oriented circles in S^3 such that c_i is once linked with c_0 and c_i, c_j are unlinked, $i, j=1, \cdots, u(\kappa)$, as shown in Fig. 1. *Let $B_1, \cdots, B_{u(\kappa)}$ be mutually disjoint $u(\kappa)$ narrow bands (2-cells) in S^3 such that B_i spans (a pair of small subarcs a_{0i} and a_i of) c_0 and c_i respectively, and*

$$B_i \cap (c_0 \cup c_1 \cup \cdots \cup c_{u(\kappa)}) = \partial B_i \cap (c_0 \cup c_i) = a_{0i} \cup a_i, \quad i=1, \cdots, u(\kappa).$$

Then

$$k = c_0 \cup c_1 \cup \cdots \cup c_{u(\kappa)} \cup \partial B_1 \cup \cdots \cup \partial B_{u(\kappa)} - (a_{01} \cup a_1) - \cdots - (a_{0,u(\kappa)} \cup a_{u(\kappa)}).$$

We will call this representative k the *canonical.*

⟨Sketch proof.⟩ From the definition of unknotting number of the knot, there must be a representative k' of κ such that if we exchange $u(\kappa)$ crossings of the regular projection of k' we have a trivial knot c_0'. Now, we can consider that the exchanging of a crossing must be done as illustrated in Fig. 2. So we have $u(\kappa)$ small circles

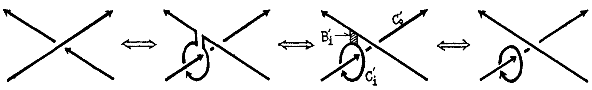

Fig. 2.

$c_1', \cdots, c_{u(\kappa)}'$ and $u(\kappa)$ bands $B_1', \cdots, B_{u(\kappa)}'$ to c_0' such that c_i' is once linked with c_0' and c_i', c_j' are unlinked and B_i' spans c_i' to c_0' (at small subarcs a_i' to a_{0i}'), $i, j=1, \cdots, u(\kappa)$, and

$$k' = c_0' \cup c_1' \cup \cdots \cup c_{u(\kappa)}' \cup \partial B_1' \cup \cdots \cup \partial B_{u(\kappa)}' - (a_{01}' \cup a_1') - \cdots - (a_{0,u(\kappa)}' \cup a_{u(\kappa)}').$$

Thus, by deforming k' isotopically into the standard position we have the canonical representative k of κ.

Remark 1. The local knot types of immersion g can be defined by the same way as embedding, that is the oriented knot type of $g(\partial \mathrm{St}(x, M^2))$ in $\partial \mathrm{St}(gx, M^4)$. Then this lemma says that for every knot $k \subset S^3 = \partial D^4$ there is a locally flat immersed 2-cell $D^2 \subset D^4$ such that

$\mathcal{J}D^2 \subset \mathcal{J}D^4$ and $k=\partial D^2 \subset S^3$. In fact, we have D^2 by joining $c_1 \cdots, c_n$ and c_0 in Fig. 1 to $n+1$ points that are chosen suitably in $\mathcal{J}D^4$. So, for any differentiable 4-manifold M^4, every homotopy class $\xi \in \pi_2(M^4)$ of maps $S^2 \to M^4$ is representable by a differentiably immersed 2-sphere.

⟨Proof of Theorem 1⟩. To complete the proof, it is sufficient to show that for any given knot type κ there exists a pair $(fS^2 \subset S^2 \times S^2)$ having the only one local knot of type κ.

To distinguish between these two 2-spheres of $S^2 \times S^2$ let's denote one of them by S_1^2 and the other S_2^2. We select a 2-cell D_i^2 in S_i^2 $(i=1,2)$, and take the 4-cell $N=D_1^2 \times D_2^2$ in $S_1^2 \times S_2^2$. In the 3-sphere $S^3=\partial N = \partial(D_1^2 \times D_2^2)$ we set the canonical representative

$$k=c_0 \cup c_1 \cup \cdots \cup c_{u(\kappa)} \cup \partial B_1 \cup \cdots \cup \partial B_{u(\kappa)} - (a_{01} \cup a_1) - \cdots - (a_{0,u(\kappa)} \cup a_{u(\kappa)})$$

of κ by Lemma 1, where $u(\kappa)$ is the unknotting number of κ, such that $c_0 \subset \partial D_1^2 \times D_2^2$ and $c_1 \cup \cdots \cup c_{u(\kappa)} \subset D_1^2 \times \partial D_2^2$. Since $\mathrm{cl}(S_i^2 - D_i^2)$ is also locally flat 2-cell in $\mathrm{cl}(S_1^2 \times S_2^2 - N)$, it is easily checked that there are mutually disjoint locally flat 2-cells $C_0^2, C_1^2, \cdots, C_{u(\kappa)}^2$ in $\mathrm{cl}(S_1^2 \times S_2^2 - N)$ such that $\partial C_i^2 = c_i$, $i=0, 1, \cdots u(\kappa)$. Thus we have a locally flat 2-cell $C_0^2 \cup B_1 \cup C_1^2 \cup \cdots \cup B_{u(\kappa)} \cup C_{u(\kappa)}^2$ in $\mathrm{cl}(S_1^2 \times S_2^2 - N)$ which bounds the knot k.

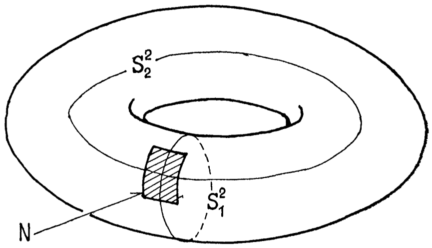

Fig. 3.

On the otherhand, we choose a vertex v in $\mathcal{J}N$. Then the join $v*k$ is a 2-cell in N and it has a locally knotted point v. The surface

$$(v*k) \cup (C_0^2 \cup B_1 \cup C_1^2 \cup \cdots \cup B_{u(\kappa)} \cup C_{u(\kappa)}^2)$$

is a 2-sphere combinatorially embedded in $S_1^2 \times S_2^2$ and it has only local knot type κ at v.

This completes the proof.

M. A. Kervaire and J. W. Milnor observed 2-spheres in $PC(2)$ in [2. p. 1654]. Now, by Lemma 1 and some elementary geometrical examinations we can state our version of their studies.

Theorem 2.

$$K(fS^2 \subset PC(2)) \supset \begin{Bmatrix} \textit{slice knot types}, \\ \langle (n, n+1)\textit{-torus knot types} \rangle, \\ \langle \textit{knot types of unknotting number } \frac{n(n-1)}{2} \rangle \end{Bmatrix},$$

where ⟨ ⟩ *denotes the cobordism class of knot types in the sense of* [1].

But we can't know that $K(fS^2 \subset PC(2)) \neq \{\text{all knot types}\}$. Thus, the following is still open. Are there 4-manifolds M^4 such that $K(fS^2 \subset M^4) \neq \{\text{slice knot types}\}$ nor $\{\text{all knot types}\}$?

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