3. *Geodesic Flows and Isotropic Flows*

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Introduction. The purpose of this paper is to give a unified treatment of geodesic flows, horocycle flows and isotropic flows. It is shown that in the two dimensional case various flows are obtained by perturbation of the geodesic flow and that the entropies of the horocycle flow, the geodesic flow and the constant isotropic flow on an n-dimensional compact Riemannian manifold with constant positive curvature are all equal to zero. Moreover by our method we can prove the theorem of Anosov to the effect that the geodesic flow on the compact Riemannian manifold with negative curvature is a C-flow and furthermore a K-flow [1]. This will be published in another paper.

1. Preparation from geometry. We shall define the maps π_* and K which we need later. For details, consult P. Dombrowski [3]. Let M be an n-dimensional differentiable Riemannian manifold and G be the fundamental tensor of M. TM_p denotes the tangent space at $p \in M$ and $TM = \bigcup_{p \in M} TM_p$ the tangent bundle. If $(x^1, \cdots, x^n)$ is a local coordinate system at $p \in M$ and $v = \sum_{i=1}^n v^i \partial / \partial x^i$, then we can take $(x^1, \cdots, x^n\ v^1, \cdots, v^n)$ as a local coordinate system at $v \in TM$. This shows that TM is a differentiable manifold, and we shall denote TM by W. For projection $\pi : W \to M$, the map $\pi_* : TW \to W$ is naturally defined. A coordinate representation of π_* is of the form $\pi_*(X_v) = \sum_{i=1}^n (\xi^i \partial / \partial x^i)$, where $TW \ni X = \sum_{i=1}^n (\xi^i \partial / \partial x^i + \xi^{i+n} \partial / \partial v^i)$, $W \ni v = \sum_{i=1}^n v^i \partial / \partial x^i$. Next, let us define the map $K : TW \to W$ as follows:

$$K(X_v) = \sum_{i,j,k=1}^n (\xi^{n+i} + \Gamma^i_{jk} \xi^j v^k) \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} are Christoffel's symbol.

By the maps π_* and K, a 2-covariant tensor $\tilde{G}$ on W is defined: $G(X, Y) = G(\pi_* X, \pi_* Y) + G(KX, KY)$, where X, Y are vector fields on W. W is a Riemannian manifold with $\tilde{G}$ as the fundamental tensor.

2. Geodesic flow. We define the geodesic flow with the following vector field S ("geodesic spray") on $W : \pi_* S_v = v$ and $KS_v = 0$, where we should consider $v \in W$ on the left side and $v \in TM$ on the right side.

Put $W_1 = \{v \in W ; \|v\| = 1\}$, then W_1 is a regular submanifold of W. We can show that W_1 is an invariant manifold of S. The geodesic flow restricted to W_1 will be called simply the *geodesic flow* on M in the following. We note that it preserves the Riemannian measure on

W.

For the geodesic spray, put $\varphi_t = \mathrm{Exp}\ tS$, $W_1 \ni v(0)$ and $TW_1 v \ni Y(0)$. Put $\varphi_t v(0) \equiv v(t)$ and $(\varphi_t)_* Y(0) \equiv Y(t)$, then we obtain the following fundamental equations.

Lemma 1. $\frac{D}{dt}\pi_* Y(t) = KY(t)$, $\left(\frac{D}{dt} = \nabla_{\pi_* S}\right)$, $\frac{D}{dt} KY(t) + R(v(t), \pi_* Y(t))v(t) = 0$, *where R is the curvature tensor.*

Using this lemma, we can prove the theorem of Anosov.

3. Horocycle flow. Let us consider the 2-dimensional case. For $v \in W$, we take the vector $u(v)$ with same length as v which is orthogonal to v. If k is a smooth function on M, then a flow is defined by the following vector field $H : \pi_* H_v = v$ and $KH_v = ku$. It can be restricted to W_1 and preserves the Riemannian measure on W_1. We will denote this restriction by $H(k)$. In particular $H(1)$ is the so-called *horocycle flow*.

Lemma 2. *If k is constant, we have*

$$D/dt\pi_* Y(t)_v = KY(t)_v, \qquad \textit{and}$$
$$D/dt\, KY(t)_v = -R(v(t), \pi_* Y(t))v(t) + kKY(t)u,$$

where $\pi_* Y_u = \pi_* Y_v$ and $G(KY_u, KY_v) = 0$.

Put $G(R(v, u)v, u) \equiv C(v)$, the sectional curvature. Using Lemma 2 we can prove.

Theorem 1. *Let M be a 2-dimensional compact Riemannian manifold. If there is $\lambda > 0$ such that $C(v) + k^2 < -\lambda^2$, then the flow $H(k)$ is a C-flow and furtheremore a K-flow.*

By the theorem of Kouchinirenko, we can prove the following theorems: Namely, we can show by computation that $\|Y(t)\|^2 = \|\pi_* Y(t)\|^2 + \|KY(t)\|^2$ has an upper estimate by some polynomial of t.

Theorem 2. *The entropy of the horocycle flow is equal to zero.*

Theorem 3. *The entropy of the geodesic flow on an n-dimensional compact Riemannian manifold with constant positive curvature is equal to zero.*

Theorems 1 and 2 together with familiar results give the following:

Theorem 4. *If M is a 2-dimensional compact Riemannian manifold with constant negative curvature -1, then for the flow $H(k)$ on M, the following three cases arise*: (1) *If $k^2 - 1 < 0$, then $H(k)$ is a C-flow and a K-flow*; (2) *If $k^2 - 1 = 0$, then $H(k)$ has zero entropy, σ-Lebesgue spectrum and ergodicity*; (3) *If $k^2 - 1 > 0$, then all orbits of $H(k)$ are closed.*

4. Isotropic flow. We will define the isotropic flows which may be considered generalization of geodesic flows and show that they can be treated analogously.

4.1. Let V_p be the set of linearly independent n-frames at $p \in M$

and $V=\bigcup_{p\in M}V_p$ Stiefel manifold $V_{0,n}$, where M is an n-dimensional compact Riemannian manifold. If $V\ni\eta=(p,v_{(a)})$, where $p=(x_1,\cdots,x_n)$, $v_{(a)}=(v^1_{(a)},\cdots,v^n_{(a)})\in TM_p$, $(a=1,2,\cdots,n)$, then we can take $(x^1,\cdots\cdots,x^n,v^1_{(1)},\cdots,v^n_{(1)},\cdots,v^1_{(n)},\cdots,v^n_{(n)})$ as the local coordinate at η. Let us define projections $\pi:V\to M$ and $P_a:TV_\eta\to TW_{v(a)}$ as follows: $\pi:(p,v_{(a)})\mapsto p$, $P_a:(\xi_0,\xi_1,\cdots,\xi_n)\mapsto(\xi_0,\xi_a)$.

Put $K_a=K\circ P_a$, then the coordinate representations of π_*, K_a are as follows:

$$\pi_*Y=\sum_{i=1}^n y^i\frac{\partial}{\partial x^i},$$
$$K_aY=\sum_{i=1}^n\Big(y^{na+i}+\sum_{j,k=1}^n\Gamma^i_{jk}y^jv^k_{(a)}\Big)\frac{\partial}{\partial x^i},$$

where $Y=(y^i,y^{na+i})\in TV_\eta$, $(a=1,\cdots,n)$.

Using π_* and K_a, 2-covariant tensor $\tilde{G}$ on V is defined as follows:

$$\tilde{G}(X,Y)=G(\pi_*X,\pi_*Y)+\sum_{a=1}^n G(K_aX,K_aY),$$

where $X,Y\in TV_\eta$, $\eta\in V$. We can easily see that V is a Riemannian manifold with $\tilde{G}$ as the fundamental tensor.

4.2. Let s, k_i $(i=1,\cdots,n-1)$ be smooth functions on M. Let us consider the flow defined by the following vector field I:

$$\pi_*I_\eta=sv_{(1)},$$
$$K_aI_\eta=-k_{a-1}v_{(a-1)}+k_av_{(a+1)},\qquad(a=1,\cdots,n)$$

where we should take $v_{(0)}=v_{(n+1)}=0$, $k_0=k_n=0$.

If $f_{ab}(\eta)=G_{ij}v^i_{(a)}v^j_{(b)}$ $(a,b=1,\cdots,n)$ and $O=\{\eta\in V;f_{ab}(\eta)=\delta_{ab}\}$, where δ_{ab} is the Kronecker δ, then O is a regular submanifold of V. We can show that O is an invariant manifold of I. We will call the flow defined by the vector field I restricted to O the *isotropic flow* and denote it by $I(s,k_1,\cdots,k_{n-1})$. Flows $I(s,k_1,\cdots,k_{n-1})$ preserve the Riemannian measure on O. If s, $k_1,\cdots,k_{n-1}$ are constant, then we will call $I(s,k_1,\cdots,k_{n-1})$ the *constant isotropic flow*.

4.3. Put $\varphi_t=\operatorname{Exp} tI$, $O\ni\eta(0)$, $TO_{\eta(0)}\ni Y(0)$. If $\varphi_t\eta(0)\equiv\eta(t)$, $(\varphi_t)_*Y(0)\equiv Y(t)$, then we can obtain an analogous fundamental equations:

Lemma 3. *$D/dt\,\pi_*Y=K_1Y$, and*

$$D/dt\,K_aY=-k_{a-1}K_{a-1}Y+k_aK_{a+1}Y-R(v_{(1)},\pi_*Y)v_{(a)}.$$

We know little about isotropic flows. We think it is interesting to investigate them. In concluding this note, we state a simple result and some conjectures.

Theorem 5. *The constant isotropic flow $I(s,k_1\cdots k_{n-1})$ on a compact Riemannian manifold with constant positive curvature has zero entropy.*

Conjecture. (1) An isotropic flow $I(1,0,0\cdots 0)$ on a compact

Riemannian manifold with negative curvature is a K-flow. (2) The above flow is not a C-flow.

References

[1] D. Anosov: Geodesic flows on a closed Riemannian manifold of negative curvature. Trudy Instituta Steklova., **90** (1967).

[2] V. I. Arnold: Some remarks on flows of line elements and frames. Sov. Math. Dokl., **2**, 562–564 (1961).

[3] P. Dombrowski: On the geometry of the tangent bundle. J. Reine und Angew. Math., **210**, 73–88 (1962).