24. On $\mathfrak{R}$-convex Sets in a Topological $\mathfrak{R}$-space

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§1. **Introduction.** In this paper we shall consider the Krein-Milman's Theorem and the applications on a topological $\mathfrak{R}$-space which has not vector spacestructure. The notion of topological $\mathfrak{R}$-spaces is introduced by E. Deák [1]-[5].

We shall first give the some definitions.

(1.1) A system R of the ordered pair (G, F) consisting of the subsets of a nonempty set X is called a *Richtung* of X, if it satisfies the following conditions:

(R_1) $(\phi, \phi), (X, X) \in R$.

(R_2) For any $(G, F) \in R$, $G \subseteqq F$ and for two different pairs $(G_1, F_1), (G_2, F_2) \in R$, $F_1 \subseteqq G_2$ or $F_2 \subseteqq G_1$.

(R_3) Let $\mathcal{G}(R)$ be a family of the first part of all elements of R. $\bigcup \{G \mid G \in \mathcal{G}^*\} \in \mathcal{G}(R)$ $(\mathcal{G}^* \subset \mathcal{G}(R), \mathcal{G}^* \neq \phi)$.

(R_4) Let $\mathcal{F}(R)$ be a family of the second part of all elements of R. $\bigcap \{F \mid F \in \mathcal{F}^*\} \in \mathcal{F}(R)$ $(\mathcal{F}^* \subset \mathcal{F}(R), \mathcal{F}^* \neq \phi)$.

(R_5) $\bigcup \{F - G \mid (G, F) \in R\} = X$.

(1.2) Let $\mathfrak{R} = \{R_\alpha \mid R_\alpha$: Richtung, $\alpha \in A\}$. A $\mathfrak{R}$-space is an ordered pair $(X, \mathfrak{R})$ such that the following separation axiom is satisfied.

(S.A) Any set of the type, $\bigcap \{F_\alpha - G_\alpha \mid (G_\alpha, F_\alpha) \in R, \alpha \in A\}$ contains at most one element.

(1.3) For a $\mathfrak{R}$-space $(X, \mathfrak{R})$, the set $G, X-F$ or $F, X-G$ is called the open or closed $\mathfrak{R}$-half spaces of X.

(1.4) A $\mathfrak{R}$-space $(X, \mathfrak{R})$ is called a *topological $\mathfrak{R}$-space* if we introduce the topology in X such that a family of all open $\mathfrak{R}$-half spaces is a subbasis.

(1.5) For any Richtung R of X, it is clear that the relation:

$(G_1, F_1) \prec (G_2, F_2) \Longleftrightarrow F_1 \subseteqq G_2$ is a linear order of R.

For any $G \in \mathcal{G}(R)$ or $F \in \mathcal{F}(R)$ is the first or second part of at most two different elements of R. $G(R:F)$ or $\bar{G}(R:F)$ denotes the smaller or larger set $G \in \mathcal{G}(R)$ such that $(G, F) \in R$, and in the same way we can define $F(R:G)$ and $\bar{F}(R:G)$ for any $G \in \mathcal{G}(R)$.

(1.6) For any nonempty set $E \subset X$ and $R \in \mathfrak{R}$ of a $\mathfrak{R}$-space $(X, \mathfrak{R})$, we give the following notations:

$$G_E(R) = \bigcup \{G \in \mathcal{G}(R) \mid G \cap E = \phi\},$$
$$F_E(R) = \bigcap \{F \in \mathcal{F}(R) \mid F \supseteqq E\},$$
$$G_x(R) = \bigcup \{G \in \mathcal{G}(R) \mid G \not\ni x\},$$

$$F_x(R)=\bigcap\{F\in\mathfrak{F}(R)\mid F\ni x\},$$
$$S_E(R)=F_E(R)-G(R:F_E(R)),$$
$$T_E(R)=\bar{F}(R:G_E(R))-G_E(R).$$

It follows from (R_5) that $(G_x(R), F_x(R))\in R$ for each $x\in X$ and each $R\in\mathfrak{R}$, and that $S_x(R)=T_x(R)$.

(1.7) Let $(X, \mathfrak{R})$ be a $\mathfrak{R}$-space and $R\in\mathfrak{R}$. If $(G, F)\in R$ and $G\subsetneqq F$, the set $F-G$ is called a R-hyperplane. The set $S\subset X$ is a $\mathfrak{R}$-hyperplane if it is a R-hyperplane for some $R\in\mathfrak{R}$. A R-hyperplane $S=F-G$ is an upper or lower R-supporting hyperplane of $E\subseteqq X$, if $S\cap E\neq\phi$ and $E\subset F$ or $E\subset X-G$. If $E-S\neq\phi$, a R-supporting hyperplane S of E is called a proper R-supporting hyperplane of E.

It is clear that the set of E has an upper or lower R-supporting hyperplane if and only if $S_E(R)\cap E\neq\phi$ or $T_E(R)\cap E\neq\phi$, and then $S_E(R)$ or $T_E(R)$ is an upper or lower R-supporting hyperplane.

We can prove that if E is a compact set, there exist the upper and lower R-supporting hyperplanes of E for any $R\in\mathfrak{R}$.

(1.8) The strong $\mathfrak{R}$-convex hull of $E\subseteqq X$ is the intersection of all $\mathfrak{R}$-halfspace containing E and we denote it by $\boldsymbol{k}(\mathfrak{R}:E)$. If $E=\boldsymbol{k}(\mathfrak{R}:E)$, E is a strong $\mathfrak{R}$-convex set. By $a\cdot\boldsymbol{k}(\mathfrak{R}:E)$ we denote the closed strong $\mathfrak{R}$-convex hull of $E\subseteqq X$ defined by the intersection of all closed $\mathfrak{R}$-halfspace containing E.

(1.9) The quasi $\mathfrak{R}$-inner of the set $E\subseteqq$ is the set

$Q(\mathfrak{R}:E)=\boldsymbol{k}(\mathfrak{R}:E)-\bigcup\{$all proper $\mathfrak{R}$-supporting hyperplane$\}$

§2. $\mathfrak{R}$-extremal sets and $\mathfrak{R}$-extremal points.

Let $(X, \mathfrak{R})$ be a $\mathfrak{R}$-space and $E\subseteqq X$. A $\mathfrak{R}$-extremal set of E is a subset $M\subseteqq E$ such that $A\subseteqq M$ whenever $A\subseteqq\boldsymbol{k}(\mathfrak{R}:E)$, $2\leqq\bar{\bar{A}}<\aleph_0$ and $M\cap Q(\mathfrak{R}:A)\neq\phi$. If $M=\{x_0\}$, x_0 is called a $\mathfrak{R}$-extremal point of E.

The following properties of $\mathfrak{R}$-extremal subsets of E can be easily verified.

(2.1) Any union of $\mathfrak{R}$-extremal subsets of a set E is a $\mathfrak{R}$-extremal subset of E.

(2.2) Any intersection of $\mathfrak{R}$-extremal subsets of a set E is a $\mathfrak{R}$-extremal subset of E.

(2.3) If A is a $\mathfrak{R}$-extremal subset of B and, B is a $\mathfrak{R}$-extremal subset of C, then A is a $\mathfrak{R}$-extremal subset of C.

(2.4) If $A\subset B\subset C$ and if A is a $\mathfrak{R}$-extremal subset of C, then A is a $\mathfrak{R}$-extremal subset of B.

(2.5) Let $\mathcal{C}$ be a family of sets in a $\mathfrak{R}$-space, and let $Y=\bigcap_{X\in\mathcal{C}}X$. If one of any two members of $\mathcal{C}$ is a $\mathfrak{R}$-extremal subset of the other, then Y is a $\mathfrak{R}$-extremal subset of each $X\in\mathcal{C}$.

§3. The Krein-Milman's Theorem.

Theorem 1. *A nonempty compact strong $\mathfrak{R}$-convex subset E of* *a topological $\mathfrak{R}$-space has at least one $\mathfrak{R}$-extremal point.*

Proof. The set E is itself a $\mathfrak{R}$-extremal subset of E. Let $\mathcal{M}$ be the totality of compact $\mathfrak{R}$-extremal subsets of E. Order $\mathcal{M}$ by the set inclusion relation. It is easy to see that if $\mathcal{M}_1$ is a linearly ordered subfamily of $\mathcal{M}$, there exists a compact $\mathfrak{R}$-extremal subset of E which is a lower bound for the subfamily $\mathcal{M}_1$.

Thus, by Zorn's lemma, $\mathcal{M}$ contains a minimal element M_0. Suppose that $\bar{M}_0 \geqq 2$. Then there exist $R \in \mathfrak{R}$ such that $\bar{F}(R: G_{M_0}(R)) \cap M_0 \subsetneqq M_0$. Since M_0 is the compact set, $M_1 = \{\bar{F}(R: G_{M_0}(R)) - G_{M_0}(R)\} \cap M_0 \neq \phi$.

On the other hand, suppose that A is a subset of $k(\mathfrak{R}: E)$ such that $Q(\mathfrak{R}: A) \cap M_1 \neq \phi$ and $2 \leqq \bar{\bar{A}} < \aleph_0$, then $A \subset M_0$, so that $A \subset X - G_{M_0}(R)$.

If $A \cap \{\bar{F}(R: G_{M_0}(R)) - G_{M_0}(R)\} = \phi$, then $A \subset X - \bar{F}(R: G_{M_0}(R))$, and therefore $Q(\mathfrak{R}: A) \cap M_1 = \phi$, which is a contradiction.

If, $A \cap \{\bar{F}(\mathfrak{R}: G_{M_0}(R)) - G_{M_0}(R)\} \neq \phi$ and $A \not\subset \{\bar{F}(R: G_{M_0}(R)) - G_{M_0}(R)\}$, then $\bar{F}(R: G_{M_0}(R)) - G_{M_0}(R)$ is a lower proper R-supporting hyperplane of A, and therefore, $Q(\mathfrak{R}: A) \cap M_1 = \phi$, which is a contradiction. Therefore $A \subset M_1 = \{\bar{F}(R: G_{M_0}(R)) - G_{M_0}(R)\} \cap M_0$ and therefore M_1 is a $\mathfrak{R}$-extremal subset of E. Since $M_1 \subset M_0$ and M_0 is a minimal $\mathfrak{R}$-extremal subset of E, it is a contradiction.

Therefore M_0 has only one point which is a $\mathfrak{R}$-extremal point of E.

Corollary. *Suppose that E is a compact strong $\mathfrak{R}$-extremal set, then for any $R \in \mathfrak{R}$, $F_E(R) - G(R: F_E(R))$ or $\bar{F}(R: G_E(R)) - G_E(R)$ has at least one $\mathfrak{R}$-extremal point of E.*

Proof. By the same way in Theorem 1, we can prove that $M = \{F_E(R) - G(R: F_E(R))\} \cap E$ is a compact $\mathfrak{R}$-extremal subset of E,

Theorem 2. *Let $\mathcal{E}(\mathfrak{R}: E)$ be all $\mathfrak{R}$-extremal points of a compact strong $\mathfrak{R}$-convex set E, then, $E = a \cdot k(\mathfrak{R}: \mathcal{E})$.*

Proof. $\mathcal{E}(\mathfrak{R}: E) \subset E$ implies $a \cdot k(\mathfrak{R}: \mathcal{E}) \subset E$. Suppose that there exists a point x_0 contained in the set $E - a \cdot k(\mathfrak{R}: \mathcal{E})$, so that there exists $R \in \mathfrak{R}$ such that $x_0 \in E$ and $x_0 \notin F_{\mathcal{E}}(R)$ or $x_0 \in E$ and $x_0 \in G_{\mathcal{E}}(R)$.

If $x_0 \in E$ and $x_0 \in F_{\mathcal{E}}(R)$, then $\{F_E(R) - G(R: F_E(R))\} \cap \mathcal{E} = \phi$.

If $x_0 \in E$ and $x_0 \in G_{\mathcal{E}}(R)$, then $\{\bar{F}(R: G_E(R)) - G_E(R)\} \cap \mathcal{E} = \phi$.

Therefore we can introduce a contradiction to the corollary of Theorem 1.

We remark that a compact strong $\mathfrak{R}$-convex set E is the strong $\mathfrak{R}$-convex hull of the set of all $\mathfrak{R}$-extremal points of E.

The following result is a generalization of a result of Jerison to a topological $\mathfrak{R}$-space.

Theorem 3. *Let $\{K_n\}$ be a sequence of compact strong $\mathfrak{R}$-convex* *sets such that $K_1 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$ and let $K = \cap K_n$. Let A_n be the set of all $\mathfrak{R}$-extremal points of K_n for each n and let A be the topological superior limit of $\{A_n\}$. Then K is the closed strong $\mathfrak{R}$-convex hull of A.*

Proof. Let F_n be the closure of $\cup_{m=n}^{\infty} A_m$, then $A = \bigcap_{n=1}^{\infty} F_n$. Since $F_n \subset K_n$ for each n, $A \subset K$ and therefore, $\cap \{F_A(R) - G_A(R) : R \in \mathfrak{R}\} \subset K$. By the Krein-Milman's Theorem in a topological $\mathfrak{R}$-space (see [5]), $k(\mathfrak{R} : F_m) = K_n$ for each n, i.e. $(G(R : F_{F_n}(R)), F_{F_n}(R)) = G(R : F_{k_n}(R))$, $F_{k_n}(R))$ and $(G_{F_n}(R), \bar{F}(R : G_{F_n}(R)) = (G_{k_n}(R), \bar{F}(R : G_{k_n}(R)))$ for each $R \in \mathfrak{R}$ and n.

Consider a fixed $R \in \mathfrak{R}$. Since F_n is a compact set, there exists a point x_n in $\{F_{F_n}(R) - G(R : F_{F_n}(R))\} \cap F_n$. The sequence $\{x_n\}$ has a cluster point x_0 which is contained in A.

Now we consider the topological superior limit of $\{F_{F_n}(R) - G(R : F_{F_n}(R))\}$ and let $X = A \cap \lim_n \sup \{F_{F_n}(R) - G(R : F_{F_n}(R))\}$. It is clear that $(G(R : F_x(R)), F_x(R)) \leqq (G(R : F_{F_n}(R)), F_{F_n}(R))$ for all n and there is no member $(G, F) \in R$ such that $(G(R : F_x(R)), F_x(R)) \lneqq (G, F) \lneqq (G(R : F_{F_n}(R)), F_{F_n}(R))$ for all n. Since $x_0 \in X$, $(G_{x_0}(R), F_{x_0}(R)) = (G(R : F_x(R)), F_x(R))$. On the other hand, since $(G(R : F_{k_n}(R))$, $F_{k_n}(R)) = (G(R : F_{F_n}(R), F_{F_n}(R))$ for all n and $K = \bigcap_{n=1}^{\infty} K_n \subset K_n$, $(G(R : F_k(R)), F_k(R)) \leqq (G(R : F_{F_n}(R)), F_{F_n}(R))$ and therefore $(G(R : F_k(R)), F_k(R)) \leqq (G_{x_0}(R), F_{x_0}(R)) \leqq (G(R : F_A(R)), F_A(R))$. Analogously we have $(G_A(R), \bar{F}(R : G_A(R))) \leqq (G_{x_0}(R), F_{x_0}(R)) \leqq (G_k(R), \bar{F}(R : G_k(R)))$.

Therefore $K \subset \cap \{F_A(R) - G_A(R) \mid R \in \mathfrak{R}\}$.

Hence we have $K = a \cdot k(\mathfrak{R} : A)$.

Theorem 4. *Let E be a compact strong $\mathfrak{R}$-convex subject of a topological $\mathfrak{R}$-space $(X, \mathfrak{R})$ and let C be the subset of E which intersects any minimal closed $\mathfrak{R}$-extremal subset of E, then E is the strong $\mathfrak{R}$-convex hull of C.*

Proof. If $k(\mathfrak{R} : C) \not\supset E$, there exists a point x_0 such that $x_0 \in E$ and $x_0 \notin k(\mathfrak{R} : C)$.

Let a R_0-halfspace $M_0 \supset C$ and $M_0 \not\ni x_0$, then $M_0 \subset G_{x_0}(R)$ or $M_0 \subset X - F_{x_0}(R)$.

If $M_0 \subset G_{x_0}(R)$, then $B = E \cap F_E(R) - G(R : F_E(R))\}$, a closed $\mathfrak{R}$-extremal subset of E, does not intersect C.

If $M_0 \subset X - F_{x_0}(R)$, then $B = E \cap \bar{F}(R : G_E(R)) - G_E(R)\}$, a closed $\mathfrak{R}$-extremal subset of E, does not intersect C. We have arrived at a contradiction.

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