23. *Lesniewski's Protothetics S1, S2. I*

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The systems S1 and S2 are defined originaly by S. Lesniewski [1]. The definitions, theorems and some relations between S1 and S2 are also shown by K. Iséki [2]. The equivalences of some laws in S1 are proved by K. Chikawa [3].

In this paper we shall prove that every theorem of S2 is a theorem of S1.

Definition. The system which has equivalence as its only primitive term, the following propositions S2A1-S2A4 as its axioms and in which are valid the rule of inference specified below, shall be called the system S2:

(a) the rule of substitution,

(b) the rule of detachment: if α and $\alpha\equiv\beta$ are both theorems of S2, then β is a theorem of S2;

(c) the rule for the distribution of a general quantifier preceeding an equivalence: if $[f, \cdots, g]\{\alpha\equiv\beta\}$ is a theorem of S2, then $[f, \cdots, g]\{\alpha\}\equiv[f, \cdots, g]\{\beta\}$ is a theorem of S2;

(d) the rule of extentionality: any equivalential law of extentionality, i.e.,

$$[f, g]\{(f\equiv g)\equiv[\varphi]\{\varphi(f)\equiv\varphi(g)\}\}$$

is a theorem of S2;

(e) the rule of definition: any correctly built definition is a theorem of S2. Of course, the definitions of S2 consist of equivalence.

S2A1 $[p, q, r]\{(p\equiv q)\equiv((r\equiv q)\equiv(p\equiv r))\}$,

S2A2 $[p, q]\{(p\equiv q)\equiv[f]\{f(p)\equiv f(q)\}\}$,

S2A3 $[p,q]\{(p\equiv q)\equiv[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\}$,

S2A4 $[f]\{f([p]\{p\})\equiv(f([p]\{p\}\equiv[p]\{p\})\equiv[q]\{f([p]\{p\})\equiv f(q)\})\}$.

Definition. The system which has implication as its only primitive term, the following proposition A1 as its axiom, and in which are valid the rule of inference specified below, shall be called the system S1;

A1 $[f, g]\{f([p]\{p\supset p\})\supset(f([p]\{p\})\supset f(q))\}$.

(a) the rule of substitution:

(b) the rule of detachment: if α and $\alpha\supset\beta$ are both theorems of that system S1, then β is a theorem of S1;

(c) the rules for the general quantifier: the first allows to add the general quantifier to the antecedent of an implication, the second to add it to the consequent of an implication, provided that the antecedent does not contain a free variable having the same form as the variable bounded by this quantifier.

(d) the rule of extentionality: given any functor of one argument and at least the second semantic order, the law of extentionality formulated for that functor, i.e.,

$$[f, p, q]\{(p\equiv q)\supset(f(p)\equiv f(q))\}$$

is a theorem of that system, where the following definitions are used:

def (1) i) $[p, q]\{(p\equiv q)\supset(((p\supset q)\supset((q\supset p)\supset[p, q]\{q\supset p\}))$
$\supset[p, q]\{(p\supset q)\supset((q\supset p)\supset[p, q]\{q\supset p\})\})\}$

ii) $[p,q]\{(((p\supset q)\supset((q\supset p)\supset[p, q]\{q\supset p\}))\supset[p, q]\{p\supset q)$
$\supset((q\supset p)\supset[p, q]\{q\supset p\})\})\supset(p\equiv q)\}$;

(e) the rule of definition: any correctly built definition is a theorem of S1. Of course, the definitions of S1 consist of implication and negation. The negation is defined as follows:

D1 i) $[p]\{\sim(p)\supset(p\supset[p]\{p\})$

ii) $[p]\{(p\supset[p]\{p\})\supset\sim(p)$.

Theorem 1. *The following axioms of Tarski-Bernays for the propositional calculus may be deduced from* A1 *in the system* S1:

T1 $[p, q, r]\{(p\supset q)\supset((q\supset r)\supset(p\supset r))\}$,

T2 $[p, q]\{q\supset(p\supset q)\}$,

T3 $[p, q]\{((p\supset q)\supset p)\supset p\}$.

For the details of the proof, see J. Slupecki [4].

Theorem 2. *The Lukasiewicz system of axioms in propositional calculus, i.e.,*

T4 $[p, q, r]\{(p\supset q)\supset((q\supset r)\supset(p\supset r))\}$,

T5 $[p]\{(\sim(p)\supset p)\supset p\}$,

T6 $[p, q]\{p\supset(\sim(p)\supset q)\}$,

result from the axiom A1 *together with the definition* D2.

Proof. T4 is equiform to T1. The proofs of T5 and T6 are given in J. Slupecki [4].

Theorem 3. *All theorems of the propositional calculus which has implication and negation as its primitive terms result from the axiom* A1 *together with the definition* D2.

Proof. All theorems of the propositional calculus result from the Lukasiewicz system of axioms in the propositional calculus. Therefore this theorem deduced directly from Theorem 2.

In the discussion below we shall use theorems of the propositional calculus, and particularly we often use the following theorems [5].

T7 $[p]\{p\supset p\}$,

T8 $[p, q]\{\sim(p\supset q)\supset p\}$, T9 $[p, q]\{\sim(p\supset\sim(q))\supset q\}$,

T10 $[p, q]\{p\supset(q\supset\sim(p\supset\sim(q)))\}$,

T11 $[p, q]\{\sim(p\supset q)\supset\sim(q)\}$.

Lemma. *The following auxiliary definition holds*

def (2) i) $[p, q]\{(p\equiv q)\supset\sim((p\supset q)\supset\sim(q\supset p))\}$,

ii) $[p, q]\{\sim((p\supset q)\supset\sim(q\supset p))\supset(p\equiv q)\}$.

Proof. Only to prove this lemma , we shall use the following abbreviations:

instead of $p\supset q$ we shall write α,

instead of $q\supset p$ we shall write β.

The abbreviation of def (1) has the form:

D2 i′) $[p, q]\{(p\equiv q)\supset((\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\})\}$,

ii′) $[p, q]\{((\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\})\supset(p\equiv q)\}$.

We shall prove further theorems of S1.

T12 $[p, q]\{((\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\})\supset\sim(\alpha\supset\sim(\beta))\}$.

Proof. (1) $((\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\})\supset$

(2) $\sim(\alpha\supset(\beta\supset\gamma))$ (D1, ii ; 1)

(3) α (T8 ; 2)

(4) $\sim(\beta\supset\gamma)$ (T11 ; 2)

(5) β (T8 ; 4)

(6) $\sim(\alpha\supset\sim(\beta))$ (T10 ; 3 ; 5)

We introduce the definition of conjunction in terms of implication and the general quantifier:

D3 i) $[p, q]\{p\cdot q\supset((p\supset(q\supset[q]\{q\}))\supset[p, q]\{p\supset(q\supset[q]\{q\})\})\}$,

ii) $[p, q]\{((p\supset(q\supset[q]\{q\}))\supset[p, q]\{p\supset(q\supset[q]\{q\})\})\supset p\cdot q\}$.

T13 $[p, q]\{\sim(\alpha\supset\sim(\beta))\supset((\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\})\}$.

Proof. (1) $\sim(\alpha\supset\sim(\beta))\supset$

(2) α (T8 ; 1)

(3) β (T9 ; 1)

(4) $(\alpha\supset(\beta\supset[p, q]\{\beta\}))\supset[p, q]\{\alpha\supset(\beta\supset[p, q]\{\beta\})\}$ (D3, i ; 2 ; 3)

(5) $(\alpha\supset(\beta\supset\gamma))\supset[p, q]\{\alpha\supset(\beta\supset\gamma)\}$ (4)

T12, T13 and def(1) show that this lemma is true.

In this paper we shall prove that every theorem of S2 is a theorem of S1. To prove this it suffices to show that the rules (a)-(e) and axioms S2A1-S2A4 of S2 may be derived from the rules and axioms of S1.

Lemma 1. *Let α and β be any propositional expressions, and let $f, \cdots, g, h$ be all the free variables of α and β. We further assume that $f, \cdots, g$ are all the free variables of α, and $f, \cdots, h$ are all the free variables of β, but we do not make any assumptions as to the semantic categories of these variables.*

If the proposition

$$[f, \cdots, g, h]\{\alpha \equiv \beta\}$$

is a theorem of S1, *then the proposition*

(1) $[f, \cdots, g]\{\alpha\} \equiv [f, \cdots, h]\{\beta\}$

is also a theorem of that system.

Proof. The proposition

T14 $[f, \cdots, g, h]\{\alpha \equiv \beta\}$

is a theorem of S1 ex hypothesi. Hence the following propositions are true.

T15 $[f, \cdots, g, h]\{\sim((\alpha \supset \beta) \supset \sim(\beta \supset \alpha))\}$ (def(2), i ; T14)

T16 $[f, \cdots, g, h]\{\alpha \supset \beta\}$ (T8 ; T15)

T17 $[f, \cdots, h]\{[f, \cdots, g]\{\alpha\} \supset \beta\}$ (T16 ; rule(c))

T18 $[f, \cdots, g]\{\alpha\} \supset [f, \cdots, h]\{\beta\}$ (T17 ; rule(c))

T19 $[f, \cdots, g, h]\{\beta \supset \alpha\}$ (T9 ; T15)

T20 $[f, \cdots, h]\{\beta\} \supset [f, \cdots, g]\{\alpha\}$ (T19 ; rule(c))

T21 $\sim(([f, \cdots, g]\{\alpha\} \supset [f, \cdots, h]\{\beta\}) \supset$
$\sim([f, \cdots, h]\{\alpha\} \supset [f, \cdots, g]\{\beta\}))$ (T10 ; T18 ; T20)

T22 $[f, \cdots, g]\{\alpha\} \equiv [f, \cdots, h]\{\beta\}$ (def(2), ii ; T21)

T22 is equiform to Proposition 1, and thus Lemma 1 is true. This lemma corresponds the rule (c) of S2.

Lemma 2. *Let α and β be any propositional expressions, and let $f, \cdots, g$ be all the free variables of α and β.*

If the proposition

(2) $[f, \cdots, g]\{\alpha \equiv \beta\}$

is added to S2 *by applying the rule of definition which is valid in that system, then this proposition is a theorem of* S1.

Proof. From the fact that Proposition 2 is added to S2 on the strength of the rule of definition valid in that system it results that the propositions

(3) $[f, \cdots, g]\{\alpha \supset \beta\}$,

(4) $[f, \cdots, g]\{\beta \supset \alpha\}$

are theorems of S1, added to the latter system on the strength of the analogous rule, valid in S1.

(5) $[f, \cdots, g]\{\sim((\alpha \supset \beta) \supset \sim(\beta \supset \alpha))\}$ (T10 ; 3 ; 4)

(6) $[f, \cdots, g]\{\alpha \equiv \beta\}$ (def(2), ii ; 5)

This proposition is equiform to (2), and thus Lemma 2 is true.

Lemma 3. *If the propositions*

(7) $\alpha \equiv \beta$,

(8) α

is a theorem of that system.

Proof. (1) $\alpha \equiv \beta$ (ex hypothesi)

(2) α (ex hypothesi)

(3) $\sim((\alpha\supset\beta)\supset\sim(\beta\supset\alpha))$ (def(2), i ; 1)
(4) $\alpha\supset\beta$ (T8 ; 3)
(5) β (4 ; 2)

This lemma corresponds to the rule of detachment of S2.

Lemma 4. *If φ is a variable functor of one argument and of the n_{th} semantic order ($n=2$), then the proposition*

(10) $[f, g]\{(f\equiv g)\equiv[\varphi]\{\varphi(f)\equiv\varphi(g)\}\}$

is a theorem of that system.

Proof. The system S1 has the following theorem i.e., the law of extentionality:

(11) $[f, p, q]\{(p\equiv q)\supset(f(p)\equiv f(q))\}$.

We add to the system the following definition;

D4 i) $[p]\{\mathrm{as}(p)\supset p\}$,
ii) $[p]\{p\supset \mathrm{as}(p)\}$.

We shall prove further theorems of S1.

T23 $[p, q]\{(p\equiv q)\supset[f]\{f(p)\equiv f(q)\}\}$ (11 ; rule(c))

T24 $[p, q]\{[f]\{f(p)\equiv f(q)\}\supset(p\equiv q)\}$

Proof. (1) $[f]\{f(p)\equiv f(q)\}\supset$
(2) $\mathrm{as}(p)\equiv\mathrm{as}(q)$ (1 ; rule(a))
(3) $\sim((\mathrm{as}(p)\supset\mathrm{as}(q))\supset\sim(\mathrm{as}(q)\supset\mathrm{as}(p)))$ (def(2), i ; 2)
(4) $\mathrm{as}(p)\supset\mathrm{as}(q)$ (T8 ; 3)
(5) $\mathrm{as}(q)\supset\mathrm{as}(p)$ (T9 ; 3)
(6) $p\supset\mathrm{as}(p)$ (D4, ii)
(7) $p\supset\mathrm{as}(q)$ (T4 ; 6 ; 4)
(8) $\mathrm{as}(q)\supset q$ (D4, ii)
(9) $p\supset q$ (T4, 7 ; 8)
(10) $q\supset\mathrm{as}(q)$ (D4, ii)
(11) $\mathrm{as}(p)\supset p$ (D4, i)
(12) $q\supset p$ (T4 ; 10 ; 5 ; 11)
(13) $\sim((p\supset q)\supset\sim(q\supset p))$ (T10 ; 9 ; 12)
(14) $p\equiv q$ (def(2), ii ; 13)

T25 $[p, q]\{\sim(((p\equiv q)\supset[f]\{f(p)\equiv f(q)\})\supset$
$\sim([f]\{f(p)\equiv f(q)\}\supset(p\equiv q)))\}$ (T10 ; T23 ; 24)

T26 $[p, q]\{(p\equiv q)\equiv[f]\{f(p)\equiv f(q)\}\}$ (def(2), ii ; T25)

Hence Lemma 4 is true. This lemma corresponds to the rule of extentionality which is valid in S2. (To be concluded.)