21. *On the Type of an Associative H-space*

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(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1969)

1. Introduction. Let X be an H-space. If the rational cohomology of X is an exterior algebra on a finite number of odd dimensional generators, then the number of such generators is called the rank of X.

The type of X is the dimensions in which the generators occur. In this paper, we obtain by the analogous method as in [3], [4], some result on an associative H-space of rank n.

Theorem. *Let X be a connected associative H-space of rank n with $H_*(X;Z)$ finitely generated as an abelian group. Let a be the generator of the rational cohomology $H^*(X;Q)$. If the degree of the generator a is $2i-1$, then we have*

$$\varphi(i)\leqq n,$$

where φ is the Euler function.

I with to express my hearty thanks to Larry Smith for suggesting this problem and giving me many valuable advices, and to Professors K. Morita and R. Nakagawa for their criticism and encouragement.

2. Some results on unstable polyalgebras.

Definition. A polynomial algebra B over the mod p Steenrod algebra A_p(p: prime) is called an unstable polyalgebra over A_p, if it is an algebra that is a left A_p-module satisfying

(1) $P_p^k x=0$ if $2k>\deg x$

(2) $P_p^k x=x^p$ if $2k=\deg x$

where we denote Sq^{2m} by P_2^m. This terminology is found in Larry Smith's paper [4].

The next theorem is fundamental in this paper.

Theorem (A. Clark [1]). *Let B be an algebra over the Steenrod algebra A_p and suppose that B is a polynomial algebra over Z_p on generators of even degree. If $2m$ is the degree of a generator of B, then B has a generator in some degree $2n$ for which $n\equiv 1-p \bmod m$, or else $m\equiv 0 \bmod p$.*

Lemma 2.1. *Let B be an unstable polyalgebra over A_p(p:odd prime) on a finite number of even dimensional generators $x_1,\cdots,x_i,\cdots$ $\cdots,x_n$ where deg $x_i=2j_i$ and $p>j_i>1$.*

Then the integer j_i satisfies one of the following conditions (A)

$$
\text{(A)}\quad
\begin{array}{ll}
j_1 \equiv 1-p & \mod j_i \\
j_2 \equiv 1-p & \mod j_i \\
\vdots & \\
j_i \equiv 1-p & \mod j_i \\
\vdots & \\
j_n \equiv 1-p & \mod j_i
\end{array}
$$

Proof. Since $j_i \not\equiv 0 \mod p$, it follows from A. Clark's Theorem that one of the conditions is clearly satisfied.

3. Some results from number theory. We need the following classical theorem in number theory.

Theorem (Dirichlet). *Every arithmetic series whose initial term and difference are relatively prime contains an infinite number of primes.*

Lemma 3.1. *Let $j_1, \cdots, j_i, \cdots, j_n$ and n be positive integers and let φ be the Euler function.*

If $\varphi(j_i) \geqq n+1$, then there is an integer s satisfying the following conditions.

$$
\begin{array}{ll}
s \not\equiv 1-j_1 & \mod j_i \\
s \not\equiv 1-j_2 & \mod j_i \\
\vdots & \\
s \not\equiv 1-j_n & \mod j_i \\
(s, j_i)=1 &
\end{array}
$$

Proof. Since $\varphi(j_i) \geqq n+1$, there are relatively different (mod j_i) integers $s_1, s_2, \cdots, s_{n+1}$
such that

$$
\begin{array}{l}
(j_i, s_1)=1 \\
(j_i, s_2)=1 \\
\vdots \\
(j_i, s_{n+1})=1
\end{array}
$$

Let $1-j_{k_1}, \cdots, 1-j_{k_m}$, $m \leqq n$ be the totality of $1-j_\nu$, $\nu=1, 2, \cdots, n$ which are relatively prime to j_i. Then there exists an s_j which is not congruent (mod j_i) to $1-j_{k_\lambda}$ for any λ with $1 \leqq \lambda \leqq m$. Such an s_j is the required integer s. This concludes the proof.

Proposition 3.2. *Let $j_1, j_2, \cdots, j_i, \cdots, j_n$ be positive integers. If for all sufficiently large primes p, one of the following conditions* (A) *are satisfied, then $\varphi(j_i) \leqq n$.*

$$
\text{(A)}\quad
\begin{array}{ll}
p \equiv 1-j_1 & \mod j_i \\
p \equiv 1-j_2 & \mod j_i \\
\vdots & \\
p \equiv 1-j_n & \mod j_i
\end{array}
$$

Proof. Suppose $\varphi(j_i) \geqq n+1$. By Lemma 3.1 there is an integer s such that

$$\begin{aligned} s &\not\equiv 1-j_1 \quad \bmod j_i \\ s &\not\equiv 1-j_2 \quad \bmod j_i \\ &\vdots \\ s &\not\equiv 1-j_n \quad \bmod j_i \\ (s, j_i) &= 1. \end{aligned}$$

By the Dirichlet's Theorem, the arithmetic series $\{a_k \mid a_k = s + j_i \cdot k\}$ contains infinitely many primes. Suppose that p is such a prime. Then we get $p \equiv s \bmod j_i$

Therefore

$$\begin{aligned} p &\not\equiv 1-j_1 \quad \bmod j_i \\ p &\not\equiv 1-j_2 \quad \bmod j_i \\ &\vdots \\ p &\not\equiv 1-j_n \quad \bmod j_i \end{aligned}$$

This concludes the proof.

4. Proof of the main theorem. Let X be a connected associative H-space of rank n with $H_*(X; Z)$ finitely generated as an abelian group. Then it follows from Hopf's Theorem that

$$H^*(X; Q) = E[x_1, \cdots, x_n]$$

where deg x_i is odd.

Since X is an associative H-space, X has the classifying space BX, as has been shown by Dold and Lashof [2]. By the generalized Borel's transgression theorem, we get for all sufficiently large prime p,

$$H^*(BX; Z_p) = P[y_1, \cdots, y_i, \cdots, y_n]$$

where $deg\ y_i = deg\ x_i + 1 = 2j_i$.

Apply Lemma 2.1 and Proposition 3.2 to this polyalgebra, and we get

$$\varphi(j_i) \leqq n.$$

This concludes the proof.

References

[1] A. Clark: On π_3 of finite dimensional H-spaces. Ann. of Math., **78**, 193–196 (1963).

[2] A. Dold and R. K. Lashof: Principal quasifibrations and fibre homotopy equivalence of bundles. Illinois J. of Math., **3**, 285–305 (1959).

[3] S. Ochiai: On the type of an associative H-space of rank three. Proc. Japan Acad., **44**, 811–815 (1968).*)

[4] L. Smith: On the type of an associative H-space of rank two (to appear in Tōhoku Math. J.).

[5] N. E. Steenrod and D. B. A. Epstein: Cohomology operations. Ann. of Math. Study No. 50. Princeton N. J.

*) Remarks on my previous paper [3]: (1) I mentioned at the footnote on page 811 that the non-existence of H-spaces with types (3, 5, 11) and (3, 11, 11) was proved by using the Steenrod operation P^2. But, later that proof was found to be incomplete. (2) In Theorem 2.1 on page 812, insert "*with even dimensional generators*" between "*polyalgebra*" and "*over*".