19. An Indirect Existence Proof of a Linear Set of the Second Category with Zero Capacity

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By giving a sufficient condition for an iterated Cantor Set to have zero capacity, Kishi-Nakai [4] gave an example of a linear set of the second category with zero capacity. This example, of course, indicates the existence of a linear set of the second category with zero measure. However the existence of the latter was already proven by K. Noshiro with an interesting indirect method (see [4]). Therefore it is desirable to give a corresponding indirect proof to the existence of the former, which is the object of the present note.

1. Before proceeding to our proof, we must recall the following two well-known results in the theory of cluster sets.

1°) *Beurling-Tsuji's theorem* [5] (see also Collingwood [3, p. 61]): A meromorphic function $f(z)$ in $|z|<1$ with

$$\iint_{|z|<1} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} r\,dr\,d\theta < \infty \quad (z=re^{i\theta}) \tag{1}$$

has an angular limit at every point in $|z|=1$ except for a possible set of capacity zero.

2°) *Collingwood's maximality theorem* [2] (see also Collingwood [3, p. 80]): For an arbitrary single-valued function $f(z)$ in $|z|<1$, the set

$$J(f)=\{e^{i\theta} \mid C_\Delta(f, e^{i\theta})=C(f, e^{i\theta}) \text{ for every } \Delta\} \tag{2}$$

is residual and hence of the second category, where $C(f, e^{i\theta})$ (resp. $C_\Delta(f, e^{i\theta})$) is the cluster set of f at $e^{i\theta}$ considered in $|z|<1$ (resp. in the Stolz angle Δ at $e^{i\theta}$).

2. Another preliminary result we need is from the theory of conformal mappings. Let $f(z)$ be the Riemann mapping function from $|z|<1$ onto a bounded simply connected region $\mathcal{D}$. Then the Carathéodory theorem asserts that $|z|=1$ corresponds to the totality of prime ends P of $\mathcal{D}$ in a one-to-one and onto fashion. The impression $I(P)$ is the intersection of the closure of regions in a determining sequence of P. Obviously

$$C(f, e^{i\theta})=I(P_\theta) \tag{3}$$

where $e^{i\theta}$ corresponds to a prime end P_θ under f. Carathéodory [1, p. 369] showed an example of $\mathcal{D}$ for which every $I(P)$ is a nondegenerate continuum. We will denote by $\mathcal{D}_C$ this particular region.

3. We are now able to give a very short indirect proof to the following

Theorem (Kishi-Nakai [4]). *There exists a residual set (the complement of a set of the first category, and hence the set of the second category) of capacity zero in the unit circle.*

Proof. Take the Carathéodory's region $\mathcal{D}_C$ (see [2]) and the Riemann mapping function f from $|z|<1$ onto $\mathcal{D}_C$. Clearly the integral in (1) is finite for f. Therefore by 1°), $C_\Delta(f, e^{i\theta})$ consists of only one point for every $e^{i\theta}$ and Δ except for a set of $e^{i\theta}$ of capacity zero. On the other hand, $C(f, e^{i\theta})=I(P_\theta)$ (see [3]) is a nondegenerate continuum (see [2]). Thus we conclude that $J(f)$ (see [2]) is of capacity zero. By 2°), we now see that $J(f)$ is the required set. Q.E.D.

References

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