17. A Generalization of Prime Ideals in Rings. II*)

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1. Recently, generalizing the notions of prime ideals and primary ideals in rings, Murata, Kurata, and Marubayashi [1] have considered the notions of f-prime ideals and f-primary ideals in rings, and obtained, along with other things, the uniqueness theorem of f-primary decompositions of ideals, under certain assumptions.

Continued from [1], in this paper, we shall investigate the ideals which can be represented as the intersection of a finite number of f-primary ideals.

Let R be an arbitrary ring. Throughout this paper, ideals will always mean two-sided ideals in R and we shall assume the following conditions as same as in [1]:

(β) For any ideal A and any ideal B not contained in $r(A)$, we have $A:B\neq\varnothing$.

(γ) If S is an f-system with kernel S^*, and if, for any ideal A, $S\cap A$ is not empty, then so is $S^*\cap A$.

(δ) For any f-primary ideal Q, we have $Q:Q=R$.

2. Isolated components

Definition 1. Let A be an ideal and let S be an f-system. The isolated component A_S of A determined by S will be defined as follows:

$$A_S=\begin{cases}\bigcup_{s\in S}(A:s) & \text{(if } S \text{ is not empty)}\\ A & \text{(if } S \text{ is empty).}\end{cases}$$

For any f-system $S\neq\varnothing$, $C(S)$ is an f-prime ideal containing $r((0))$. If $s\in S$, then $s\notin r((0))$ and hence by the assumption (β) we have $(0):s\neq\varnothing$. This shows that $A:s$ and whence A_S is not empty. So, it can be proved similarly as in [1] that A_S is an ideal containing A.

Another characterization of f-primary ideals can be given by means of isolated components.

Proposition 2. *An ideal Q is f-primary if and only if, for any f-system S, either $Q_S=Q$ or $Q_S=R$ holds.*

Proof. Suppose that Q is f-primary. If S is empty, then the assertion is trivial. Now we may suppose that there exists a non-

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empty f-system S such that $Q_S \neq Q$. Let b be an element such as $b \in Q_S$ and $b \notin Q$. Then there exists at least one $s \in S$ such that $f(s)f(b) \subseteqq Q$. Since Q is f-primary, we have $s \in r(Q)$, and thus we can choose an element $a \in S \cap Q$. Since $Q : a = R$ by the assumption (δ), we have $Q_S = R$.

Conversely, let us suppose that, for any f-system S, either $Q_S = Q$ or $Q_S = R$ holds and that Q is not f-primary. Then there exist $b \notin Q$ and $c \notin r(Q)$ such that $f(c)f(b) \subseteqq Q$. Since $c \notin r(Q)$, for some f-prime ideal P we have $Q \subseteqq P$ and $c \notin P$. If we set $S = C(P)$, then S is an f-system and $b \in Q_S$. Therefore $Q \subsetneqq Q_S$. It follows from the assumption that $Q_S = R$ and hence there exists at least one $s \in S$ such that $f(s)f(c) \subseteqq Q$. Since P is an f-prime ideal containing Q, we have, by [1, Lemma 1.4], either $s \in P$ or $c \in P$, which is impossible in any case.

If an ideal A has an f-primary decomposition, then the isolated component of A can be expressed in terms of its f-primary components:

Theorem 3. *Let A be an ideal, and let S be an f-system. Suppose that $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, where each Q_i is an f-primary ideal. If $r(Q_i)$ meet S for $m+1 \leqq i \leqq n$ but not for $1 \leqq i \leqq m$, then we have*

$$A_S = Q_1 \cap Q_2 \cap \cdots \cap Q_m.$$

If $r(Q_i)$ meet S for $1 \leqq i \leqq n$, then $A_S = R$.

Proof. If S is empty, then the assertion is trivial. We may therefore assume that S is not empty. Let $x \in A_S$. Then, for some $s \in S$, we have $f(s)f(x) \subseteqq A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. Consequently, if $1 \leqq i \leqq m$, since $s \notin r(Q_i)$, we have $x \in Q_i$, and hence $A_S \subseteqq Q_1 \cap Q_2 \cap \cdots \cap Q_m$.

For $m+1 \leqq j \leqq n$, $r(Q_j) \cap S$ is not empty and hence so is $Q_j \cap S$ and also, by the assumption (γ), so is $Q_j \cap S^*$. Since S^* is an m-system, for $s_{m+1} \in Q_{m+1} \cap S^*$ and $s_{m+2} \in Q_{m+2} \cap S^*$, there exists $z \in R$ such that $s'_{m+2} = s_{m+1} z s_{m+2} \in Q_{m+1} \cap Q_{m+2} \cap S^*$. Similarly, there exists $z' \in R$ such that $s'_{m+3} = s'_{m+2} z' s_{m+3} \in Q_{m+1} \cap Q_{m+2} \cap Q_{m+3} \cap S^*$ for $s_{m+3} \in Q_{m+3} \cap S^*$. Continuing in this way, we obtain after a finite number of steps an element s'_n which is in $Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n \cap S^*$. Thus it follows from the assumption (δ) that $Q_j : s'_n = R$ for $m+1 \leqq j \leqq n$ and hence $(Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n) : s'_n = R$. On the other hand, we have $Q_1 \cap Q_2 \cap \cdots \cap Q_m \subseteqq (Q_1 \cap Q_2 \cap \cdots \cap Q_m) : s'_n$. Therefore we have $Q_1 \cap Q_2 \cap \cdots \cap Q_m \subseteqq A : s'_n \subseteqq A_S$.

If, for $1 \leqq i \leqq n$, $r(Q_i)$ meet S, then the above proof shows that there exists an element $s'_n \in Q_1 \cap Q_2 \cap \cdots \cap Q_n \cap S^*$ which satisfies that $(Q_1 \cap Q_2 \cap \cdots \cap Q_n) : s'_n = R$. Thus we have $A_S = R$. This completes the proof.

Let S be an f-system. Combining this theorem with Proposition 2, we obtain that if Q is f-primary, then Q_S is R or Q according as $r(Q)$ meets or does not meet S.

From Theorem 3, we see at once

Corollary 4. *A decomposable ideal has at most a finite number of isolated components.*

3. Isolated set

Lemma 5. *Suppose that an ideal A has an f-primary decomposition: $A=Q_1\cap Q_2\cap\cdots\cap Q_n$. Then any f-prime ideal P which contains A must contain at least one of the Q_i.*

Proof. If $P=R$, then the assertion is trivial, and so we may suppose that there exists an f-prime ideal $P\neq R$ such that $A\subseteqq P$ and $Q_i\not\subseteqq P$ for $1\leqq i\leqq n$. If we put $S=C(P)$, then S is an f-system and $S\cap Q_i$ and whence $S\cap r(Q_i)$ is not empty for all i. We have, by Theorem 3, $A_S=R$. Thus, for any element $x\in R$, there exists some element $s\in S$ such that $f(s)f(x)\subseteqq A$ holds. This implies, by [1, Lemma 1.4], that $x\in P$, a contradiction.

Suppose that an ideal A has an f-primary decomposition, and let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be its normal decomposition. Then as is seen from [1, Theorem 3.7], the number of f-primary components and the radicals of f-primary components depend only on A and not on the particular normal decomposition considered.

Definition 6. A subset $\{r(Q_1), r(Q_2), \cdots, r(Q_m)\}$ of the radicals is called an isolated set of A, if for $m+1\leqq j\leqq n$, each $r(Q_j)$ is not contained in any of $r(Q_i)$ for $1\leqq i\leqq m$.

Proposition 7. *Suppose that an ideal A has an f-primary decomposition. Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be its normal decomposition, and let $r(Q_i)=\bigcap_k P_{ik}$ be the expression of $r(Q_i)$ as the intersection of all the minimal f-prime ideals belonging to Q_i. Then the following conditions are equivalent:*

(1) *The set $\{r(Q_1), r(Q_2), \cdots, r(Q_m)\}$ is an isolated set of A,*

(2) *For each Q_i, $1\leqq i\leqq m$, there exists at least one minimal f-prime ideal $P_{ik_i}=P_i^*$ such that P_i^* does not contain P_{jk} for all j, $m+1\leqq j\leqq n$, and for all k,*

(3) *Each $r(Q_i)$, $1\leqq i\leqq m$, does not contain the intersection $Q_{m+1}\cap Q_{m+2}\cap\cdots\cap Q_n$.*

We come now to the second uniqueness theorem for normal decompositions:

Theorem 8. *Suppose that an ideal A has an f-primary decomposition, and let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be its normal decomposition. If $\{r(Q_1), r(Q_2), \cdots, r(Q_m)\}$ is an isolated set of A, then $Q_1\cap Q_2\cap\cdots\cap Q_m$ depends only on $r(Q_1), r(Q_2), \cdots, r(Q_m)$ and not on the particular normal decomposition of A.*

Proof. Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n=Q_1'\cap Q_2'\cap\cdots\cap Q_n'$ be two normal decompositions of A such that $r(Q_i)=r(Q_i')$ for $1\leqq i\leqq n$. If we denote $Q_{m+1}\cap Q_{m+2}\cap\cdots\cap Q_n$ and $Q'_{m+1}\cap Q'_{m+2}\cap\cdots\cap Q'_n$ by Q and Q' respectively, then by Proposition 7, (3), Q is not contained in any of $r(Q_i)$ for $1\leqq i\leqq m$, and hence it follows from [1, Proposition 3.5] that $Q_i:Q=Q_i$ and also $Q'_i:Q=Q'_i$ for $1\leqq i\leqq m$. But on the other hand, since $Q_j\supseteqq Q$ for $m+1\leqq j\leqq n$, by the assumption (δ) we have $R=Q_j:Q_j\subseteqq Q_j:Q$ and hence $Q_j:Q=R$. These relations show that $Q_1\cap Q_2\cap\cdots\cap Q_m=A:Q=Q'_1\cap Q'_2\cap\cdots\cap Q'_m\cap(Q':Q)$. Thus we have $Q_1\cap Q_2\cap\cdots\cap Q_m\subseteqq Q'_1\cap Q'_2\cap\cdots\cap Q'_m$, and similarly we have $Q'_1\cap Q'_2\cap\cdots\cap Q'_m\subseteqq Q_1\cap Q_2\cap\cdots\cap Q_m$, which completes the proof.

Remark. It follows from Proposition 7,(2) that for each Q_i, $1\leqq i\leqq m$, there exists at least one minimal f-prime ideal P_i^* belonging to Q_i such that P_i^* does not contain P_{jk} for all j, $m+1\leqq j\leqq n$, and for all k. Since any f-prime ideal containing an ideal contains a minimal f-prime ideal belonging to it, for $m+1\leqq j\leqq n$ each Q_j is not contained in any of P_i^* for $1\leqq i\leqq m$. Theorem 3 then shows that, for $1\leqq i\leqq m$, each $A_{P_i^*}$ can be expressed as the intersection of certain of $Q_1, Q_2, \cdots, Q_m$, one of which is certainly Q_i. It follows that we have

$$Q_1\cap Q_2\cap\cdots\cap Q_m=A_{P_1^*}\cap A_{P_2^*}\cap\cdots\cap A_{P_m^*}.$$

Since each minimal element of the set $\{r(Q_1), r(Q_2), \cdots, r(Q_n)\}$ form on its own an isolated set of A, we see at once

Corollary 9. *Let $r(Q)$ be a minimal element in the set $\{r(Q_1), r(Q_2), \cdots, r(Q_n)\}$ of the radicals of the f-primary components of A. Then the f-primary component corresponding to $r(Q)$ is the same for all normal decompositions of A.*

Reference

[1] K. Murata, Y. Kurata, and H. Marubayashi: A generalization of prime ideals in rings. I (to appear in Osaka J. Math.).