34. *Modular Pairs in Atomistic Lattices with the Covering Property*

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1. Introduction. In the previous paper [4], a lattice L is called a DAC-lattice when both L and its dual are atomistic lattices with the covering property. The lattice $\mathcal{L}$ of closed subspaces of a linear system, appeared in Mackey [2], is an example of a DAC-lattice. In [2; p. 168], Mackey proved that a pair of elements of $\mathcal{L}$ is both modular and dual-modular if and only if it is stable modular. In this paper we shall show (Theorem 2) that this statement can be proved in general DAC-lattices. As a consequence of this result, we shall obtain a condition on a DAC-lattice which is equivalent to cross-symmetry. In the last section, we shall show some results on cross-symmetry of the lattice of closed subspaces of a locally convex space.

2. Symmetry of modular relations. Let a and b be elements of a lattice. We say that (a, b) is a *modular pair* (resp. a *dual-modular pair*) and write $(a, b)M$ (resp. $(a, b)M^*$) when

$$(c\vee a)\wedge b=c\vee(a\wedge b) \quad \text{for every} \quad c\leqq b$$

(resp. $$(c\wedge a)\vee b=c\wedge(a\vee b) \quad \text{for every} \quad c\geqq b).$$

(Note that $(a, b)M^*$ is equivalent to $(b, a)M^*$ in the sense of [4].)

A lattice L is called *M-symmetric* (resp. *M*-symmetric*) when $(a, b)M$ implies $(b, a)M$ (resp. $(a, b)M^*$ implies $(b, a)M^*$) in L. L is called *cross-symmetric* (resp. *dual cross-symmetric*) when $(a, b)M$ implies $(b, a)M^*$ (resp. $(a, b)M^*$ implies $(b, a)M$) in L.

Lemma 1. *Let a, b and c be elements of a lattice L.*

(i) *If $(a, b)M$ and $(a\wedge b, c)M$ then $(a_1, b\wedge c)M$ for any element a_1 of the interval $L[a\wedge c, a]$.*

(ii) *If $(a, b)M$ then $(a_1, b_1)M$ for any $a_1\in L[a\wedge b, a]$ and $b_1\in L[a\wedge b, b]$.*

Proof. (i) Let $a\wedge c\leqq a_1\leqq a$. Then $a_1\wedge c=a\wedge c$. If $d\leqq b\wedge c$, then by $(a, b)M$ and $(a\wedge b, c)M$ we have

$$(d\vee a_1)\wedge(b\wedge c)\leqq(d\vee a)\wedge b\wedge c=\{d\vee(a\wedge b)\}\wedge c$$
$$=d\vee(a\wedge b\wedge c)=d\vee(a_1\wedge b\wedge c)\leqq(d\vee a_1)\wedge(b\wedge c).$$

Hence $(a_1, b\wedge c)M$.

(ii) Assume $(a, b)M$ and let $a\wedge b\leqq b_1\leqq b$. Since $(a\wedge b, b_1)M$, it follows from (i) that

$$(a_1, b_1)M \quad \text{for any} \quad a_1\in L[a\wedge b_1, a]=L[a\wedge b, a].$$

The following theorem is due to Schreiner (a generalization of [6], Theorem 6).

Theorem 1. *Any cross-symmetric lattice is M-symmetric. Any dual cross-symmetric lattice is M^*-symmetric.*

Proof. It is evident that

(1) $(a, b)M$ in a lattice L if and only if $(a, b)M$ in $L[a\wedge b, a\vee b]$.

Assume that L is cross-symmetric and let $(a, b)M$ in L. For any $c\in L[a\wedge b, a]$, since $(c, b)M$ by Lemma 1, we have $(b, c)M^*$ by the assumption. Hence

$$(c\vee b)\wedge a = a\wedge(b\vee c) = (a\wedge b)\vee c = c\vee(b\wedge a).$$

Therefore $(b, a)M$ in $L[a\wedge b, a\vee b]$, and hence $(b, a)M$ in L by (1). The second statement holds by duality.

3. Modularity in DAC-lattices. A subset S of a lattice L is called *join-dense* in L when

$$a = \vee(x\in S\,;\, x\leqq a) \quad \text{for every} \quad a\in L.$$

In a lattice L, we write $a\prec b$ when $a<b$ and there does not exist $c\in L$ with $a<c<b$.

Let L be a lattice with 0. An element $a\in L$ is called an *atom* when $0\prec a$, and a is called finite when it is the join of a finite number of atoms. L is called *finite-modular* when $(b, a)M$ for any finite element $a\in L$ and for any $b\in L$. L is called *atomistic* when the set of all atoms is join-dense in L. The following property of L is called the *covering property*:

If p is an atom and $p\nleqq a$ then $a\prec a\vee p$.

An atomistic lattice with the covering property is called an *AC-lattice*. A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual L^* are AC-lattices.

By [3], Lemma 4, any finite-modular AC-lattice is M^*-symmetric, and by [4], Theorem 2.1, any DAC-lattice is finite-modular, M-symmetric and M^*-symmetric.

Lemma 2. *In a lattice, if $(a, b)M$, $(c, a\vee b)M$ and $c\wedge(a\vee b)\leqq a$ then $(c\vee a, b)M$ and $(c\vee a)\wedge b = a\wedge b$.*

Proof. Wilcox [7], Lemma 1.2.

Lemma 3. *Let S be a join-dense set in a lattice L, and let $a, b\in L$. If $(a, b\vee x)M$ for every $x\in S$ with $x\nleqq b$ then $(a, b)M^*$.*

Proof. Let $c\geqq b$. Evidently, $(c\wedge a)\vee b\leqq c\wedge(a\vee b)$. Let $x\in S$ and $x\leqq c\wedge(a\vee b)$. We shall prove that $x\leqq(c\wedge a)\vee b$. This is evident when $x\leqq b$. When $x\nleqq b$, we have $(a, b\vee x)M$ by the assumption. Hence

$$x\leqq(b\vee a)\wedge(b\vee x) = b\vee\{a\wedge(b\vee x)\}\leqq b\vee(a\wedge c) = (c\wedge a)\vee b.$$

Since S is join-dense, we have $c\wedge(a\vee b)\leqq(c\wedge a)\vee b$.

Lemma 4. *Let a and b be elements of an AC-lattice L. If*

$(a, x)M$ for every $x \succ b$ then $(a, b)M^*$.

Proof. The set S of all atoms of L is join-dense. If $x \in S$ and $x \nleqq b$ then $b \prec b \vee x$ by the covering property. Hence this lemma is a consequence of Lemma 3.

Lemma 5. *In a finite-modular AC-lattice L, if $(a, b)M^*$ then $(a \vee x, b \vee y)M^*$ for all finite elements x and y.*

Proof. By the dual property of Lemma 1 (i),

(1) $(a, b)M^*$ and $(a \vee b, c)M^*$ together imply $(a, b \vee c)M^*$.

Let $(a, b)M^*$. If y is a finite element, then since $(a \vee b, y)M^*$ by [4], Lemma 2.2 (ii), we have $(a, b \vee y)M^*$ by (1). Similarly, since L is M^*-symmetric, $(a, b \vee y)M^*$ implies $(a \vee x, b \vee y)M^*$ for every finite element x.

Lemma 6. *Let a and b be elements of a finite-modular AC-lattice L. If $(a, b)M$ and $(a, b)M^*$ then $(a, b_1)M^*$ for any $b_1 \in L[a \wedge b, b]$.*

Proof. It follows from [3], Lemma 4 that $(a, b)M^*$ is equivalent to the following $(a \neq 0,\ b \neq 0)$:

If p is an atom with $p \leqq a \vee b$ then there exist atoms q and r such that $p \leqq q \vee r$, $q \leqq a$ and $r \leqq b$.

Assume $(a, b)M$ and $(a, b)M^*$, and let $a \wedge b \leqq b_1 \leqq b$. We may assume $a \neq 0$ and $b_1 \neq 0$. Let p be an atom with $p \leqq a \vee b_1$. It suffices to show that there exist atoms q and r such that $p \leqq q \vee r$, $q \leqq a$ and $r \leqq b_1$. Since $p \leqq a \vee b$ and $(a, b)M^*$, there exist atoms q_1 and r_1 such that $p \leqq q_1 \vee r_1$, $q_1 \leqq a$ and $r_1 \leqq b$. When $p = q_1$, then $q = q_1$ and any atom $r \leqq x$ may be used. When $p \neq q_1$, by the covering property we have $p \vee q_1 = q_1 \vee r_1$, whence $r_1 \leqq p \vee q_1 \leqq a \vee b_1$. Since $(a, b)M$, we have

$$r_1 \leqq (b_1 \vee a) \wedge b = b_1 \vee (a \wedge b) = b_1.$$

Hence $q = q_1$ and $r = r_1$ have the desired property.

Theorem 2. *Let a and b be elements of a DAC-lattice L. The following three statements are equivalent.*

(α) *$(a, b)M$ and $(a, b)M^*$.*

(β) *$(a, x)M$ for every $x \succ b$.*

(γ) *$(a, x)M^*$ for every $x \prec b$.*

Proof. (i) We shall prove that (γ) implies (α). We may assume $b \neq 0$. Since L^* is an AC-lattice, there exists an element c with $c \prec b$ in L. Then there exists an atom p such that $b = c \vee p$. Since $(a, c)M^*$ by (γ), we have $(a, b)M^*$ by Lemma 5. Moreover, by (γ), in L^* we have $(a, x)M$ for every $x \succ b$. Hence, by Lemma 4, we have $(a, b)M^*$ in L^*, whence $(a, b)M$ in L. Therefore (γ) implies (α).

If (β) holds, then (γ) holds in L^* and hence (α) holds in L^*. Therefore (α) holds in L also.

(ii) We shall prove that (α) implies (β). Let $x \succ b$. When $x \leqq a \vee b$, then in L^* we have $a \wedge b \leqq x \leqq b$. Hence, by Lemma 6, ($\alpha$)

implies $(a, x)M^*$ in L^*, whence $(a, x)M$ in L. When $x \nleq a \vee b$, we take an atom p such that $x = b \vee p$. Then $p \wedge (b \vee a) = 0$, since otherwise we would have $x \leq a \vee b$. Since L is M-symmetric, we have $(b, a)M$ and $(p, b \vee a)M$. Moreover, $p \wedge (b \vee a) = 0 \leq b$. Hence by Lemma 2, we have $(p \vee b, a)M$. Therefore (α) implies (β).

By the duality, (α) implies (γ) also.

Corollary. *Let L be a DAC-lattice* (hence L is M-symmetric and M^*-symmetric). *L is cross-symmetric if and only if in L*

$$(a, b)M \text{ implies } (a, c)M \text{ for any } c \succ b.$$

L is dual cross-symmetric if and only if in L

$$(a, b)M^* \text{ implies } (a, c)M^* \text{ for any } c \prec b.$$

Proof. If $(a, b)M$, then by the equivalence of (α) and (β) in Theorem 2, $(a, b)M^*$ is equivalent to $(a, c)M$ for every $c \succ b$.

Remark 1. It follows from this corollary that if a DAC-lattice L is cross-symmetric then, in L, $(a, b)M$ implies $(a \vee x, b \vee y)M$ for all finite elements x and y. Compare with Lemma 5.

4. The lattice of closed subspaces of a locally convex space. Let E be a locally convex space. The set $L_c(E)$ of all closed subspaces of E forms an irreducible complete DAC-lattice by [4], Corollary 1 of Theorem 6.1. It was proved by Mackey ([2], pp. 166–167) that a pair (A, B) in $L_c(E)$ is dual-modular if and only if the linear sum $A+B$ is closed in E and that (A, B) is modular if and only if the mapping $\varphi : (x, y) \rightarrow x+y$ of $A \times B$ into E is a weak homomorphism (a homomorphism for weak topologies).

If E is metrisable, then since both the domain and the range of φ are Mackey spaces, φ is a weak homomorphism if and only if it is a homomorphism (see [5], p. 159). If E is a Fréchet space (metrisable and complete), then by Banach's homomorphism theorem, φ is a homomorphism if and only if its range $A+B$ is closed (see [5], p. 77). Therefore we obtain the following:

Theorem 3. *If E is a Fréchet space then $L_c(E)$ is cross-symmetric and dual cross-symmetric* (hence $(A, B)M$, $(B, A)M$, $(A, B)M^*$ and $(B, A)M^*$ are all equivalent).

Remark 2. This theorem is a generalization of Theorem III-13 in Mackey [2]. In [2; p. 173], he showed existence of an incomplete normed space E such that $L_c(E)$ is neither cross-symmetric nor dual cross-symmetric.

Remark 3. Let E be an inner product space. The following three statements are equivalent.

(α) E is complete (E is a Hilbert space).

(β) $L_c(E)$ is cross-symmetric.

(γ) $L_c(E)$ is cross-symmetric and dual cross-symmetric.

The implication $(\alpha) \Rightarrow (\gamma)$ follows from Theorem 3, and $(\gamma) \Rightarrow (\beta)$ is trivial. The implication $(\beta) \Rightarrow (\alpha)$ was proved by Holland [1].

Question. Is there a normed space E such that $L_c(E)$ is dual cross-symmetric but not cross-symmetric?

References

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