60. *Structure Theorems for Some Classes of Operators*

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1. We consider bounded linear operators on a Hilbert space H. Denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$ the spectrum, the point spectrum, the residual spectrum and the continuous spectrum respectively, by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ the spectral radius and by $W(T)=\{(Tx, x) : \|x\|=1\}$ the numerical range. It is known [3] that $W(T)$ is convex and conv $\sigma(T) \subset \mathrm{cl}\, W(T)$ (conv = convex hull, cl = closure). An operator T is said to be hyponormal if $T^*T - TT^* \geqslant 0$, or equivalently if $\|T^*x\| \leq \|Tx\|$ for every $x \in H$. As in [1] an operator is said to be restriction-convexoid (reduction-convexoid) if the restriction of T to every invariant (invariant under T and T^*) subspace is convexoid, where convexoid means that conv $\sigma(T) = \mathrm{cl}\, W(T)$.

In this Note we give some theorems on structure of hyponormal and restriction-convexoid operators whose spectrum lies on a convex curve.

2. Our main result in this section is

Theorem 1. *If T is a hyponormal operator and has the following properties*

1° $T^p = ST^{*p}S^{-1} + C$ *for some S for which $o \bar{\in} \mathrm{cl}\, W(S)$ and C = compact operator*

2° *if* $\mu, \lambda \in \sigma(T)$, $1 + \frac{\lambda}{\bar{\mu}} + \left(\frac{\lambda}{\bar{\mu}}\right)^2 + \cdots + \left(\frac{\lambda}{\bar{\mu}}\right)^{p-1} \neq o$

then T is a normal operator.

For the proof we need the following

Lemma 1. *If T is a hyponormal operator which is the sum of a self-adjoint operator A and a compact operator C, then T is a normal operator.*

Proof. Since T is hyponormal it is known [10] that T can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on a product space $H = H_1 \oplus H_2$ where H_1 is spanned by all the proper vectors of T such that: (a) T_1 is normal and $\sigma(T_1) = \mathrm{cl}\, \sigma_p(T)$, (b) T_2 is hyponormal and $\sigma_p(T_2) = \varnothing$, (c) T is normal if and only if T_2 is normal.

From the fact that $T = A + C$ we conclude by Lemma 2 [10] that $\sigma_c(T_2) \subset \sigma(A)$ and therefore $\sigma_c(T_2)$ is real. Since $\sigma_r(T)$ is open [9] and (T) is closed, we have that $\partial_r(T_2) \subset \sigma_p(T_2) \cup \sigma_c(T_2) = \sigma_c(T_2)$ (∂ = boundary). Therefore T_2 is selfadjoint since T_2 is hyponormal with real spectrum.

Proof of the theorem. If $I(H)$ is the ideal of compact operators and $\omega(T)$, the Weyl spectrum (this means $\omega(T)=\bigcap_{C\in I(H)}\sigma(T+C)$) we obtain by the same reason as in [4] that $\omega(T)$ is real. By a result of Coburn [2] we conclude that $\sigma(T)=\omega(T)\cup\sigma_0(T)$ where $\sigma_0(T)$ contains only isolated eigenvalues of finite multiplicity. Let T_1 the restriction of T to the space H_1 generated by eigenvectors corresponding to eigenvalues $\lambda\in\sigma_0(T)$. If we denote $H_2=H_1^{\perp}$, we obtain

$$H=H_1\oplus H_2$$

and if $C=T_1\oplus 0$ and $A=0\oplus T_2$ we obtain $T=A+C$ where A is selfadjoint and C is compact (with finite range) and by Lemma 1 it follows that T is a normal operator.

Corollary. *If T is a hyponormal operator with compact imaginary part, then T is normal.*

It is easy to see that for every operator we have

$$T=T^*+2i\,\mathrm{Im}\,T$$

and by Theorem 1 for $p=1$ the corollary follows.

Another proof of this corollary is in [7] and [10]

3. Theorem 2. *If a reduction-convexoid operator T whose spectrum lies on a convex curve is the sum of a compact operator C and a generalized nilpotent operator Q then T is normal.*

Proof. Since T is convexoid and $\sigma(T)$ lies on a convex curve, T can be expressed as a direct sum $T_1\oplus T_2$ defined on a product space $H_1\oplus H_2$, where H_1 is spanned by all the eigenvectors of T, such that T_1 is normal with $\sigma(T_1)=\mathrm{cl}\,\sigma_p(T)$. By Weyl's Theorem [3 problem 143] we conclude that $\sigma(T)\subset\sigma(Q)=\{0\}$ except the eigenvalues which implies $\sigma(T_2)\subset\{0\}$ since $\sigma(T_2)=\sigma_c(T_2)\subseteq\sigma_c(T)$. By Lemma 6 [6], H_1 reduces T and thus T_2 is convexoid operator with a single point in the spectrum. Since this point is zero we conclude that $T_2=0$ which implies $H_2=0$ and T is normal.

We recall that an operator T, $\|T\|\leq 1$ and $\sigma(T)\subset\{z: |z|=1\}$ is called unimodular contraction.

Corollary 1. *If T is a convexoid unimodular contraction and $T=C+Q$ then T is unitary.*

Corollary 2. *If a reduction-convexoid operator T with compact imaginary part has the spectrum on a convex curve, then T is normal.*

Proof. By Weyl's Theorem $\sigma(T)$ is real except the eigenvalues and we conclude as above that $\sigma(T_2)$ is real and convexoid. Therefore T_2 is selfadjoint and by Theorem 2 [6] T is normal.

Theorem 3. *If T has the following properties:*

1° *spectral (in the sense of Dunford)*

2° *is restriction-convexoid with compact imaginary part, then there exists a direct decomposition of H, $H=H_\infty+H_1+H_2+\cdots$ such* *that*

a) H_i, $i=1, 2, \cdots$ *is invariant under* T

b) $T|_{H_\infty}$ *is scalar* ($T|_{H_\infty}$ *is the restriction of* T *to* H_∞)

c) $T|_{H_i}=\mu_i I$, $i=1, 2, \cdots$, μ_i *complex numbers.*

Proof. Since T is almost normal [8], it follows that there exists a direct decomposition of H with properties a) and b) and

$$T|_{H_i}=\mu_i I+Q_i,$$

Q_i are compact nilpotent operators.

But $T|_{H_i}$ is convexoid and therefore $T|_{H_i}-\mu_i I$ is also convexoid with a single point (zero) in the spectrum. Then

$$T|_{H_i}-\mu_i I=0$$

Theorem 4. *If a restriction-convexoid operator whose spectrum lies on a convex curve is polynomially compact, then* T *is normal.*

Proof. By Theorem 2 [6] we have that $T=T_1+T_2$ as above with the same properties. Since H_2 is invariant under T and T is polynomially compact then $\sigma_c(T|_{H_2})\subseteq\{\lambda : p(\lambda)\neq o\}$ where $p(.)$ is a polynomial with $p(T)=$compact. Therefore $\sigma(T_2)$ is a finite set and thus $H_2=\{o\}$. Indeed, in the contrary case, since T is restriction convexoid we have that $\sigma(T_2)=\sigma_p(T_2)$ which is a contradiction.

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