# 58. Leśniewski's Protothetics S1, S2. III

By Shôtarô TANAKA

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1969)

In this paper, we shall prove that every theorem of S2 is a theorem of S1.

**Lemma 8.** *The proposition*

(16) $[f]\{f(0)\equiv(f(1)\equiv[q]\{f(0)\equiv f(q)\})\}$

*is a theorem of* S1.

**Proof.** We add to S1 the following definitions:

D12 $[f, p]\{\psi_1\langle f\rangle(p)\equiv(f(0)\supset f(p))\}$,

D13 $[f, p]\{\psi_2\langle f\rangle(p)\equiv(f(p)\supset f(0))\}$.

T84 $[f]\{f(0)\cdot f(1)\supset[q]\{f(0)\supset f(q)\}\}$

Proof.
(1) $f(0)$
(2) $f(1)\supset$
(3) $f(0)\supset f(1)$ (T2 ; 1)
(4) $f(0)\supset f(0)$ (T2 ; 2)
(5) $\psi_1\langle f\rangle(1)$ (D12 ; 3)
(6) $\psi_1\langle f\rangle(0)$ (D12 ; 4)
(7) $[q]\{\psi_1\langle f\rangle(q)\}$ (A1′ ; 6 ; 5)
(8) $[q]\{f(0)\supset f(q)\}$ (D12 ; 7)

T85 $[f]\{f(0)\cdot f(1)\supset[q]\{f(q)\supset f(0)\}\}$

Proof.
(1) $f(0)$
(2) $f(1)\supset$
(3) $f(1)\supset f(0)$ (T2 ; 1)
(4) $f(0)\supset f(0)$ (T2 ; 2)
(5) $\psi_2\langle f\rangle(1)$ (D13 ; 3)
(6) $\psi_2\langle f\rangle(0)$ (D13 ; 4)
(7) $[q]\{\psi_2\langle f\rangle(q)\}$ (A1′ ; 5 ; 6)
(8) $[q]\{f(q)\supset f(0)\}$ (D13 ; 7)

T86 $[f]\{f(0)\cdot f(1)\supset[q]\{f(0)\equiv f(q)\}\}$ (T84 ; T85 ; T10 ; def (2), ii)

T87 $[f]\{f(0)\supset(f(1)\supset[q]\{f(0)\equiv f(q)\})\}$ (T86)

T88 $[f]\{f(0)\cdot[q]\{f(0)\equiv f(q)\}\supset f(1)\}$

Proof.
(1) $f(0)$
(2) $[q]\{f(0)\equiv f(q)\}\supset$
(3) $f(0)\equiv f(1)$ (2 ; rule(a))
(4) $f(0)\supset f(1)$ (def (2) ; T8 ; 3)
(5) $f(1)$ (4 ; 1)

T89 $[f]\{f(0)\supset([q]\{f(0)\equiv f(q)\}\supset f(1))\}$ (T88)

T90 $[f]\{f(0)\supset(f(1)\equiv[q]\{f(0)\equiv f(q)\})\}$ (def (2), ii ; T87 ; T89)

T91 $[f]\{(f(1)\equiv[q]\{f(0)\equiv f(q)\})\supset f(0)\}$

**Proof.** (1) $f(1)\equiv[q]\{f(0)\equiv f(q)\}\supset$

(2) $[q]\{f(0)\equiv f(q)\}\supset f(1)$ (def (2), i; T9)

(3) $(f(0)\equiv f(0))\supset f(1)$ (2; rule (a))

(4) $f(1)$ (3)

(5) $[q]\{f(0)\equiv f(q)\}$ (1; 4)

(6) $f(0)\equiv f(1)$ (5)

(7) $f(1)\supset f(0)$ (def (2), ii; T9)

(8) $f(0)$ (7; 4)

T92 $[f]\{f(0)\equiv(f(1)\equiv[q]\{f(0)\equiv f(q)\})\}$ (def (2), i; T90; T91)

T92 is equiform to the proposition (16), and thus Lemma 8 is true. We have used terms '0' and '1' instead of abbreviations of the propositions '$[p]\{p\}$' and '$[p]\{p\supset p\}$', respectively. Now we add to S1 the following definitions:

def (3) $1\equiv[p]\{p\supset p\}$,

def (4) $0\equiv[p]\{p\}$.

We shall prove further theorems of *S*1.

T93 $[p]\{p\}\equiv[p]\{p\}$ (T10; def (2), ii)

T94 $([p]\{p\}\equiv[p]\{p\})\supset 1$

**Proof.** (1) $([p]\{p\}\equiv[p]\{p\})\supset[p]\{p\supset p\}$ (T2)

(2) $[p]\{p\supset p\}\supset 1$ (def (3))

(3) $([p]\{p\}\equiv[p]\{p\})\supset 1$ (T1; 1; 2)

T95 $1\equiv([p]\{p\}\equiv[p]\{p\})$

**Proof.** (1) $1\supset([p]\{p\}\equiv[p]\{p\})$ (T2; T93)

(2) $([p]\{p\}\equiv[p]\{p\})\supset 1$ (T94)

(3) $1\equiv([p]\{p\}\equiv[p]\{p\})$ (T10; 1; 2; def (2), ii)

**Lemma 9.** *The proposition*

(17) $[f]\{f([p]\{p\})\equiv(f([p]\{p\}\equiv[p]\{p\})\equiv[q]\{f([p]\{p\})\equiv f(q)\})\}$

*is a theorem of* S1.

**Proof.** We shall prove further theorems.

T96 $[f]\{f([p]\{p\})\equiv(f(1)\equiv[q]\{f([p]\{p\})\equiv f(q)\})\}$

**Proof.** (1) $0\equiv[p]\{p\}$ (def (4))

(2) $[f]\{f(0)\equiv f([p]\{p\})\}$ (rule (*d*); 1)

(3) $(f(0)\equiv(f(1)\equiv[q]\{f(0)\equiv f(q)\}))\equiv(f([p]\{p\})$
$\equiv(f(1)\equiv[q]\{f([p]\{p\})\equiv f(q)\}))$ (2)

(4) $f([p]\{p\})\equiv(f(1)\equiv[q]\{f(0)\equiv f(q)\})$ (3; T92)

To obtain (3) we use functional substitution:

$f(r)$: $f(r)\equiv(f(1)\equiv[q]\{f(r)\equiv f(q)\})$.

This substitution is possible on the strength of auxiliary definitions.

T97 $[f]\{f([p]\{p\})\equiv(f([p]\{p\}\equiv[p]\{p\})\equiv[q]\{f([p]\{p\})\equiv f(q)\})\}$

**Proof.** (1) $1\equiv([p]\{p\}\equiv[p]\{p\})$ (T95)

(2) $[f]\{f(1)\equiv f([p]\{p\}\equiv [p]\{p\})\}$ (rule (*d*); 1)

(3) $(f([p]\{p\})\equiv (f(1)\equiv [q]\{f([p]\{p\})\equiv f(q)\}))\equiv (f([p]\{p\})$
$\equiv (f([p]\{p\}\equiv [p]\{p\})\equiv [q]\{f([p]\{p\})\equiv f(q)\}))$
(2)

(4) $f([p]\{p\})\equiv (f([p]\{p\}\equiv [p]\{p\})\equiv [q]\{f([p]\{p\})\equiv f(q)\})$
(3; T96)

To obtain (3), we have used functional substitution:

$f(r): f([p]\{p\})\equiv (f(r)\equiv [q]\{f([p]\{p\})\equiv f(q)\})$.

Such a substitution is possible on the strength of auxiliary definition.

T97 is equiform to (17), and thus Lemma 9 is true. T97 is equiform to axiom S2A4.

**Theorem 4.** *Every theorem of* S2 *is a theorem of* S1.

**Proof.** It follows from Lemmata 4, 5, 7, 9 and T26 that axioms of S2 are theorems of S1. It follows from Lemmata 1, 2, 3, 4 that the rule of S2 may be derived from the rules and the axiom of S1. Therefore, Theorem 4 is true.

## References


[1] S. Leśniewski: Grundzüge eines neuen Systems der Grundlagen der Mathematik. Fundamenta Mathematicae, **65**, 1–81 Warszawa (1929).

[2] K. Iséki: Symbolic Logic Propositional Calculi (in Japanese), Vol. 1, Maki publisher (1968).

[3] K. Chikawa: On equivalences of laws in elementary protothetics. I. Proc. Japan Acad., **43**, 743–747 (1967).

[4] J. Slupecki: St. Leśniewski's protothetics. Studia Logica, **1**, 54–55 (1953).

[5] J. Lukasiewicz: Elements of Mathematical Logic (in Polish), authorised notes from lectures prepared by M. Presburger, Warszawa (1929).