57. *Leśniewski's Protothetics S1, S2. II*

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In this paper, finally we shall prove that every theorem of S2 is a theorem of S1.

For further theorems we shall use the following abbreviations:

instead of $[p]\{p\equiv p\}$ we shall write 1,

instead of $[p]\{p\}$ we shall write 0.

The abbreviation of A1 has the form:

A1′ $[f, q]\{f(1)\supset(f(0)\supset f(q))\}$.

Lemma 5. *The proposition*

(12) $[p, q, r]\{(p\equiv q)\equiv((r\equiv q)\equiv(p\equiv r))\}$

is a theorem of S1.

Proof. We shall prove further theorems of S1.

T27	$1\supset 1$	
T28	$0\supset 0$	
T29	$1\equiv 1$	(T10; T27; def (2), ii)
T30	$0\equiv 0$	(T10; T28; def (2), ii)
T31	$(0\equiv 1)\supset(1\equiv 0)$	

Proof.

(1)	$(0\equiv 1)\supset$	
(2)	$\sim((0\supset 1)\supset\sim(1\supset 0))$	(def (2), i; 1)
(3)	$1\supset 0$	(T9; 2)
(4)	$0\supset 1$	(T8; 2)
(5)	$\sim((1\supset 0)\supset\sim(0\supset 1))$	(T10; 3; 4)
(6)	$1\equiv 0$	(def (2), ii; 5)

T32	$(1\equiv 0)\supset(0\equiv 1)$	(likewise T31)
T33	$(0\equiv 1)\equiv(1\equiv 0)$	(T10; T31; T32; def (2), ii)
T34	$(1\equiv 0)\equiv(0\equiv 1)$	(T10; T32; T31; def (2), ii)
T35	$(1\equiv 1)\equiv(1\equiv 1)$	(T10; T29; def (2), ii)
T36	$(0\equiv 0)\equiv(0\equiv 0)$	(T10; T30; def (2), ii)
T37	$(1\equiv 1)\equiv((0\equiv 1)\equiv(1\equiv 0))$	(T10; T29; T33; def (2), ii)
T38	$(1\equiv 1)\equiv((1\equiv 1)\equiv(1\equiv 1))$	(T10; T29; T35; def (2), ii)

From the fact that Lemma 2 is true, we may introduce the following definition.

D5	$[p]\{\varphi_1(p)\equiv((1\equiv 1)\equiv((p\equiv 1)\equiv(1\equiv p)))\}$	
T39	$\varphi_1(0)$	(T37; D5)
T40	$\varphi_1(1)$	(T38; D5)
T41	$[r]\{\varphi_1(r)\}$	(A1′; T40; T39)

T42 $[r]\{(1\equiv 1)\equiv((r\equiv 1)\equiv(1\equiv r))\}$ (D5; T41)

T43 $(1\equiv 0)\supset((0\equiv 0)\equiv(1\equiv 0))$

Proof. (1) $(1\equiv 0)\supset$

(2) $(0\equiv 0)\supset(1\equiv 0)$ (T2; 1)

(3) $(1\equiv 0)\supset(0\equiv 0)$ (T2; T30)

(4) $\sim(((0\equiv 0)\supset(1\equiv 0))\supset\sim((1\equiv 0)\supset(0\equiv 0)))$ (T10; 2; 3)

(5) $(0\equiv 0)\equiv(1\equiv 0)$ (def (2), ii; 4)

T44 $((0\equiv 0)\equiv(1\equiv 0))\supset(1\equiv 0)$

Proof. (1) $((0\equiv 0)\equiv(1\equiv 0))\supset$

(2) $\sim(((0\equiv 0)\supset(1\equiv 0))\supset\sim((1\equiv 0)\supset(0\equiv 0)))$ (def (2), i; 1)

(3) $(0\equiv 0)\supset(1\equiv 0)$ (T8; 2)

(4) $1\equiv 0$ (3; T30)

T45 $(1\equiv 0)\equiv((0\equiv 0)\equiv(1\equiv 0))$ (T10; T43; T44)

T46 $(1\equiv 0)\equiv((1\equiv 0)\equiv(1\equiv 1))$ (likewise T45)

D6 $[p]\{\varphi_2(p)\equiv((1\equiv 0)\equiv((p\equiv 0)\equiv(1\equiv p))\}$

T47 $\varphi_2(0)$ (D6; T45)

T48 $\varphi_2(1)$ (D6; T46)

T49 $[r]\{\varphi_2(r)\}$ (A1′; T47; T48)

T50 $[r]\{(1\equiv 0)\equiv((r\equiv 0)\equiv(1\equiv r))\}$ (D6; T49)

D7 $[p,r]\{\varphi_3\langle r\rangle(p)\equiv((1\equiv p)\equiv((r\equiv p)\equiv(1\equiv r))\}$

T51 $\varphi_3\langle r\rangle(1)$ (D7; T42)

T52 $\varphi_3\langle r\rangle(0)$ (D7; T50)

T53 $[q,r]\{\varphi_3\langle r\rangle(q)\}$ (A1′; T52; T51)

T54 $[q,r]\{(1\equiv q)\equiv((r\equiv q)\equiv(1\equiv r))\}$ (D7; T53)

T55 $(0\equiv 0)\equiv((0\equiv 0)\equiv(0\equiv 0))$

T56 $(0\equiv 0)\equiv((1\equiv 0)\equiv(0\equiv 1))$

D8 $[p]\{\varphi_4(p)\equiv((0\equiv 0)\equiv((p\equiv 0)\equiv(0\equiv p)))\}$

T57 $\varphi_4(0)$ (D8; T55)

T58 $\varphi_4(1)$ (D8; T56)

T59 $[r]\{\varphi_4(r)\}$ (A1′; T57; T58)

T60 $[r]\{(0\equiv 0)\equiv((r\equiv 0)\equiv(0\equiv r))\}$ (D8; T59)

T61 $[r]\{(0\equiv 1)\equiv((r\equiv 1)\equiv(0\equiv r))\}$

D9 $[q,r]\{\varphi_5\langle r\rangle(q)\equiv((0\equiv q)\equiv((r\equiv q)\equiv(0\equiv r))\}$

T62 $[r]\{\varphi_5\langle r\rangle(0)\}$ (D9; T60)

T63 $[r]\{\varphi_5\langle r\rangle(1)\}$ (D9; T61)

T64 $[q,r]\{\varphi_5\langle r\rangle(q)\}$ (A1′; T62; T63)

T65 $[q,r]\{(0\equiv q)\equiv((r\equiv q)\equiv(0\equiv r))\}$

D10 $[p,q,r]\{\varphi_6\Leftarrow q,r\Rightarrow(p)\equiv((p\equiv q)\equiv((r\equiv q)\equiv(p\equiv r)))\}$

T66 $[q,r]\{\varphi_6\Leftarrow q,r\Rightarrow(0)\}$ (D10; T56)

T67 $[q,r]\{\varphi_6\Leftarrow q,r\Rightarrow(1)\}$ (D10; T54)

T68 $[p, q, r]\{\varphi_6 \Leftarrow q, r \Rightarrow (p)\}$ (A1′; T66; T67)

T69 $[p, q, r]\{(p\equiv q)\equiv((r\equiv q)\equiv(p\equiv r))\}$ (D10; T68)

T69 is equiform to the proposition (12), and thus Lemma 5 is true. T69 is also equiform to the axiom S2A1.

T70 $[p, q]\{(p\equiv q)\equiv(q\equiv p)\}$ (T69; rule (a), (b))

We shall prove further theorems of S1. We add to S1 the following definition, which contains terms other than primitive terms. This definition is considered as an abbreviation of D3.

D11 $[p, q]\{p\cdot q\equiv\sim(p\supset\sim(q))\}$

T71 $[p, q, r]\{(p\equiv(q\supset r))\cdot(r\supset q)\cdot p\supset(q\equiv r)\}$

Proof. (1) $p\equiv(q\supset r)$

(2) $r\supset q$

(3) $p\supset$

(4) $\sim((p\supset(q\supset r)\supset\sim((q\supset r)\supset p))$ (def (2), i; 1)

(5) $p\supset(q\supset r)$ (T8; 4)

(6) $q\supset r$ (5; 3)

(7) $\sim((q\supset r)\supset\sim(r\supset q))$ (T10; 6; 2)

(8) $q\equiv r$ (def (2), ii; 7)

T72 $[p, q, r]\{(p\equiv(q\supset r)\cdot(r\supset q))\supset(p\supset(q\equiv r))\}$ (T39)

T73 $[p, q, r]\{(p\equiv(q\supset r))\cdot(q\equiv r)\supset p\}$

Proof. (1) $p\equiv(q\supset r)$

(2) $(q\equiv r)\supset$

(3) $\sim((p\supset(q\supset r))\supset\sim((q\supset r)\supset p))$ (def (2), i; 1)

(4) $(q\supset r)\supset p$ (T9; 3)

(5) $\sim((q\supset r)\supset\sim(r\supset q))$ (def (2), i; 2)

(6) $q\supset r$ (T8; 5)

(7) p (4; 6)

T74 $[p, q, r]\{(p\equiv(q\supset r)\cdot(r\supset q))\supset((q\equiv r)\supset p)\}$ (T72)

T75 $[p, q, r]\{(p\equiv(q\supset r)\cdot(r\supset q)\}\supset(p\equiv(q\equiv r))\}$ (T72; T74)

Lemma 6. *If the proposition*

(13) $[p, \cdots]\{\omega\}$

is a theorem of S1, *then the proposition*

(14) $[p, \cdots, r]\{r\equiv(\omega\equiv r)\}$

is also a theorem of the system, where ω *is a propositional expression.*

Proof. The following propositions are theorems of S1.

T76 $[p, \cdots, r]\{r\supset(\omega\supset r)\}$ (T2)

T77 $[p, \cdots, r]\{r\supset(r\supset\omega)\}$ (T2; (13))

T78 $[p, \cdots, r]\{r\supset(\omega\equiv r)\}$ (T10; T76; T77; def (2), ii)

T79 $[p, \cdots, r]\{(\omega\equiv r)\supset r\}$ (def (2), i; T8; (13))

T80 $[p, \cdots, r]\{r\equiv(\omega\equiv r)\}$ (T10; T78; T79; def (2), ii)

T80 is equiform to the proposition (14), and thus Lemma 6 is true.

Lemma 7. *The proposition*

(15) $[p, q]\{(p\equiv q)\equiv[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\}$

is a theorem of S1.

T81 $[p, q]\{(p\equiv q)\supset[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\}$

Proof. (1) $(p\equiv q)\supset$

(2) $(p\equiv q)\equiv[f]\{f(p)\equiv f(q)\}$ (T25)

(3) $[f]\{f(p)\equiv f(q)\}$ (2; 1; Lemma 3)

(4) $((p\equiv q)\equiv[f]\{f(p)\equiv f(q)\})\supset[f]\{f(p)\equiv f(q)\}$ (T2; 3)

(5) $[f]\{f(p)\equiv f(q)\}\supset((p\equiv q)\equiv[f]\{f(p)\equiv f(q)\})$ (T2; 2)

(6) $\sim(((p\equiv q)\equiv[f]\{f(p)\equiv f(q)\})\supset[f]\{f(p)\equiv f(q)\})$
$\supset\sim([f]\{f(p)\equiv f(q)\}\supset((p\equiv q)\equiv[f]\{f(p)\equiv f(q)\}))$ (T10; 4; 5)

(7) $((p\equiv q)\equiv[f]\{f(p)\equiv f(q)\})\equiv[f]\{f(p)\equiv f(q)\}$ (def (2), ii; 6)

(8) $[r]\{r\equiv((p\equiv q)\equiv r)\}$ (Lemma 6; 1)

(9) $[f]\{(f(p)\equiv f(q))\equiv((p\equiv q)\equiv(f(p)\equiv f(q)))\}$ (8)

(10) $[f]\{f(p)\equiv f(q)\}\equiv[f]\{(p\equiv q)\equiv(f(p)\equiv f(q))\}$ (9; Lemma 1)

(11) $[f]\{(p\equiv q)\equiv(f(p)\equiv f(q))\}$ (10; 3)

(12) $[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}$ (T70; 11)

We add to S1 the following definition:

D11 $[p]\{vr(p)\equiv(p\supset p)\}$

T82 $[p, q]\{[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\supset(p\equiv q)\}$

Proof. (1) $[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\supset$

(2) $(vr(p)\equiv vr(q))\equiv(p\equiv q)$ (1; rule (a))

(3) $vr(p)\equiv vr(q)$ (D11; T10; def (2), ii)

(4) $p\equiv q$ (2; 3)

T83 $[p, q]\{(p\equiv q)\equiv[f]\{(f(p)\equiv f(q))\equiv(p\equiv q)\}\}$ (T10; T81; def (2), ii; T82)

T83 is equiform to the proposition (15); and thus Lemma 7 is true. This proposition is also equiform to S2A3, which is one of axioms of S1.