55. On Limit Spaces and the Double Weak Limit. I

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§ 1. **Introduction.** **1.0.** Our purpose is to construct the limit spaces (i.e. generalized topological spaces [2] p. 273) J_w, J_{pw} and J_{sep} defined on the set $\tilde{J}$ shown in **1.2** which characterize the generalized double weak limits (itself or with the restriction on sign) expressed by filter. These spaces J_w, J_{pw}, J_{sep} and another space $J_\wedge$ also show the difference among the conditions which characterize the (topological) limit space.

1.1. Let E be a set. Let τx (by τ) be the set of filters defined on the set E corresponding to $x \in E$. We show here the following properties of $\tau x\,(L^1) \sim (L^4)$ [2] p. 273, [3] pp. 451–452.

(L^1) τx for any $x \in E$ is a $\wedge$ ideal. Here $\wedge$ ideal is the set of filters satisfying the following conditions (i) (ii);

(i) $\mathfrak{F}_1 \cap \mathfrak{F}_2 \equiv \{F \cup G\,;\, F \in (\mathfrak{F}_1),\, G \in (\mathfrak{F}_2)\} \in \tau x$ for any $\mathfrak{F}_1, \mathfrak{F}_2 \in \tau x$,

(ii) all filters $\mathfrak{F}$ finer than $\mathfrak{F}_1 \in \tau x$ (i.e. $(\mathfrak{F}) \supseteq (\mathfrak{F}_1)$ holds) are also the elements of τx. Here $(\mathfrak{F}_1)$, $(\mathfrak{F}_2)$ and $(\mathfrak{F})$ are the sets consisting of the elements of $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}$ respectively.

Hereafter let $[x]$ denote the filter with the base $\{x\}$, and let $[\mathfrak{B}(x)]$ denote the weakest filter in τx (if it exists).

(L^2) τx for any $x \in E$ contains $[x]$.

(L^3) τx for any $x \in E$ contains $[\mathfrak{B}(x)]$.

(L^4) Corresponding to a $V \in [\mathfrak{B}(x)]$ there exists an element $W(\subseteq V)$ of $[\mathfrak{B}(x)]$ such that $V \in [\mathfrak{B}(y)]$ holds for all $y \in W$.

If τ satisfies (L^1) (L^2), (E, τ) is called a limit space [2] p. 273. If τ satisfies $(L^1) \sim (L^3)$, (E, τ) is called a principal ideal limit space. If τ satisfies $(L^1) \sim (L^4)$, (E, τ) is called a topological space. Limit space is L space by M. Frechet described by the filter. The following (T_1) (T_2) are the axioms of separation in limit space. $(T_1)[x] \bar{\in} \tau y$ holds for any two distinct elements x, y in E. (T_2) $\tau x \cap \tau y = \phi$ holds for any two distinct elements x, y in E.

Let (E, τ) be a limit space. If $\mathfrak{F} \in \tau x$, we call that $\mathfrak{F}$ tends to $x \in E$ by τ, and that x is the limit from $\mathfrak{F}$ by τ. If $[\{x_i\,;\, i \geqslant n\}\,;\, x_i \in E]$ becomes the base of a filter $\mathfrak{F} \in \tau x$, we say that $\{x_n\}$ tends to x by τ. Let A be a set in E. $\bar{A}$ (the closure of A) consists of the points $x \in E$ such that there exists a filter $\mathfrak{F} \in \tau x$ satisfying $F \cap A \neq 0$ for any $F \in (\mathfrak{F})$.

The purpose of the theory on limit space is to construct the limit on E independently of the set theory.

1.2. Let $u_n \in L^2_{(-\infty,\infty)}$. If $\lim_{n\to\infty}\int u_n^2\varphi dx$ is finite and definite for any fixed $\varphi(x)\in B$ (B; the space of real valued uniformly almost periodic functions of x, where x is a real variable; $-\infty<x<+\infty$), we say that this $L^2_{(-\infty,\infty)}$-function's sequence $\{u_n\}$ has double weak limit [4] p. 139 denoted by $d.w.B.\ \lim_{n\to\infty} u_n$. Let J denote the set consisting of the real valued $L^2_{(-\infty,\infty)}$-function's sequences with double weak limit.

Let $\tilde{O}\equiv[\{f_n\};\ \lim_{n\to\infty}\int f_n^2\varphi dx=0$ for $\forall\varphi(x)\in B]\subseteq J$. Since $\tilde{O}$ is a vector space (Lemma I–3), the equivalent class $\tilde{\mathfrak{f}}$ of $\{f_n\}\in J$ is defined by $[\{g_n\};\ \{f_n-g_n\}\in\tilde{O},\ \{g_n\}\in J]$. The set consisting of the equivalent classes in J is denoted by $\tilde{J}$. O and $\mathfrak{f}$ denote the classes $\tilde{O}$ and $\tilde{\mathfrak{f}}$ regarded as the point in $\tilde{J}$. Let L_2 be the set consisting of the equivalent classes $\tilde{f}$ (or f) $\equiv[\{g_n\};\ \{f-g_n\}\in\tilde{O},\ f\in L^2_{(-\infty,\infty)},\ \{g_n\}\in J]$. $\tilde{f}$ (or f) can be regarded as the function contained in $L^2_{(-\infty,\infty)}$, and L_2 can be regarded as $L^2_{(-\infty,\infty)}$. The corresponding convergence in $\tilde{J}$ to the one by original double weak limit is the one for the sequence $\{u_n\}$ with the terms contained in L_2 to $\mathfrak{u}\in\tilde{J}$. Namely $d.w.B.\ \lim_{n\to\infty} u_n(=u)$ becomes $\tilde{J}\ni\mathfrak{u}\equiv \mathrm{cl}[\{u_n;\ u_n\in L_2\}]$. Furthermore, this convergence $d.w.B.\ \lim_{n\to\infty} u_n=\mathfrak{u}$ can be extended to the one for the sequence with the terms contained in $\tilde{J}$ to an element in $\tilde{J}$. D. Judge defines the original double weak convergence (for the sequence with the terms in $L^2_{(-\infty,\infty)}$) in order to construct a generalized Hilbert space containing $\delta^{\frac{1}{2}}$ and $\nu^{\frac{1}{2}}$ by the meaning of sequence [5] p. 378 which is the direct product $L^2_{(-\infty,\infty)}\otimes\prod_{-\infty<s<\infty}\{a_s\delta^{\frac{1}{2}}(x-s)\}\otimes\prod_{-\infty<t<\infty}\{b_t\nu^{\frac{1}{2}}\exp itx\}$ with the norm $\|\sum_{\nu=1}^{\infty}a_\nu e_\nu+\sum\tilde{a}_s\delta^{\frac{1}{2}}(x-s)+\sum b_t\nu^{\frac{1}{2}}\exp(itx)\|^2=\sum_{\nu=1}^{\infty}|a_\nu|^2+\sum|\tilde{a}_s|^2+\sum|b_t|^2$, where $\{e_\nu;\ \nu=1,2,\cdots<\infty\}$ is a complete orthonormal system in $L^2_{(-\infty,\infty)}$. Here ν is a functional (by Y. Takahashi and by H. Umezawa) satisfying $\int\nu(x)\varphi(x)dx=\lim_{T\to\infty}1/(2T)\cdot\int_{-T}^{T}\varphi(x)dx$ for any fixed $\varphi\in B$.

1.3. Let's show here the equivalent relation in J by using $\tilde{O}$ in §2. Example I-1 in §3 shows the $\wedge$ ideal not to be limit space and not relating to double weak limit. The weakest filter base of $\tilde{\tau}x(x\in E)$ in Example I-1 is the family of the sets constructed by the elimination of x from the elements of a given filter ($\neq[x]$).

§2. The equivalent relation of the sequences in J.

Let J denote the space consisting of the real valued $L^2_{(-\infty,\infty)}$-function's sequences with double-weak limit, and $\tilde{O}$ denote the zero class $\left[\{f_n\};\ \lim_{n\to\infty}\int f_n^2\varphi dx=0 \text{ for } \forall\varphi\in B,\ f_n\in L^2_{(-\infty,\infty)}\right]$.

Lemma I-1. *If $\varphi \in B$ (the space of real valued uniformly almost periodic functions), then φ^2, $|\varphi|$, $\varphi^+ \equiv (\varphi + |\varphi|)/2$ and $\varphi^- \equiv (\varphi - |\varphi|)/2$ are also contained in B.*

Proof. Since $\varphi \in B$ is bounded and continuous [6] p. 86, φ^2, $|\varphi|$, φ^+, φ^- are bounded and continuous.

Let $l_\varphi(\varepsilon)$ be the number dependent on φ associated to $\varepsilon > 0$ satisfying $|\varphi(x+T) - \varphi(x)| < \varepsilon$ for a given $T \in [a, a + l_\varphi(\varepsilon)]$ for any real a. Since the above numbers for φ^2, $|\varphi|$, φ^+ and φ^- associated to $\varepsilon > 0$ become $l_{\varphi^2}(\varepsilon) = l_\varphi(\varepsilon/\{2\,\mathrm{Max}\,(1, \sup|\varphi|)\})$ and $l_{|\varphi|}(\varepsilon) = l_{\varphi^+}(\varepsilon) = l_{\varphi^-}(\varepsilon) = l_\varphi(\varepsilon)$, then φ^2, $|\varphi|$, φ^+ and φ^- are also contained in B [6] p. 93.

Lemma I-2. *If $\{f_n\}$, $\{g_n\}$ are the elements in J, there exists a constant $K > 0$ (independent of φ) satisfying $\left|\int f_n \cdot g_n \cdot \varphi dx\right| \leqslant K \sup|\varphi|$ for any $\varphi \in B$.*

Proof. Since 1 is the element of B, sequences $\left\{\int f_n^2 dx\right\}$ and $\left\{\int g_n^2 dx\right\}$ are convergent. Then $\left|\int f_n \cdot g_n \cdot \varphi dx\right| \leqslant \int |f_n| \cdot |g_n| \cdot |\varphi| dx \leqslant \left\{\int f_n^2 \cdot |\varphi| dx + \int g_n^2 \cdot |\varphi| dx\right\} \Big/ 2 \leqslant \left\{\int f_n^2 dx + \int g_n^2 dx\right\} \Big/ 2 \cdot \sup|\varphi| \leqslant K \sup|\varphi|$ holds for any $\varphi \in B$, where K is a constant independent of φ and n.

Let $\{f_n\}$ and $\{g_n\}$ be the elements in J. $\{f_n\} \pm \{g_n\} \equiv \{f_n \pm g_n\}$ and $k\{f_n\} \equiv \{kf_n\}$.

Lemma I-3. *$\tilde{O}$ becomes a vector space contained in J.*

Proof. (i) Since $\varphi^\pm \in B$ holds for any $\varphi \in B$ (Lemma I-1), $\lim_{n\to\infty}\int f_n^2 \cdot \varphi^\pm dx = 0$ holds for any $\varphi \in B$ provided that $\lim_{n\to\infty}\int f_n^2 \cdot \varphi dx = 0$ holds for any $\varphi \in B$. Since $\int f_n^2 \varphi dx = \int f_n^2 \varphi^+ dx + \int f_n^2 \varphi^- dx$ holds, $\lim_{n\to\infty}\int f_n^2 \varphi dx = 0$ holds for any $\varphi \in B$ provided that $\lim_{n\to\infty}\int f_n^2 \varphi^\pm dx = 0$ hold for any $\varphi \in B$. Then, if and only if $\lim_{n\to\infty}\int f_n^2 \varphi dx = 0$ holds for any $\varphi \in B$, $\lim_{n\to\infty}\int f_n^2 \varphi^\pm dx = 0$ holds for any $\varphi \in B$.

(ii) If $\lim_{n\to\infty}\int f_n^2 \varphi dx = \lim_{n\to\infty}\int g_n^2 \varphi dx = 0$ holds for any $\varphi \in B$, $\lim_{n\to\infty}\int f_n^2 \varphi^\pm dx = \lim_{n\to\infty}\int g_n^2 \varphi^\pm dx = 0$ holds for any $\varphi \in B$. Since $0 \leqslant \int (f_n + g_n)^2 \varphi^+ dx \leqslant 2\left[\int f_n^2 \varphi^+ dx + \int g_n^2 \varphi^+ dx\right]$ and $0 \geqslant \int (f_n + g_n)^2 \varphi^- dx \geqslant 2\left[\int f_n^2 \varphi^- dx + \int g_n^2 \varphi^- dx\right]$ hold, $\lim_{n\to\infty}\int (f_n + g_n)^2 \varphi^\pm dx = 0$ holds. Then $\lim_{n\to\infty}\int (f_n + g_n)^2 \varphi dx = 0$ holds. Namely if $\{f_n\}$, $\{g_n\} \in \tilde{O}$, $\{f_n + g_n\} \in \tilde{O}$.

(iii) Furthermore, if $\lim_{n\to\infty}\int f_n^2 \varphi dx = 0$, $\lim_{n\to\infty}\int (kf_n)^2 \varphi dx = \lim_{n\to\infty} k^2 \int f_n^2 \varphi dx = 0$ holds.

(iv) Then $\tilde{O}$ becomes a vector space contained in J.

Lemma I-4. *Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be the elements in J, and let $\tilde{\mathfrak{f}}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ be $[\{u_n\}; \{f_n-u_n\} \in \tilde{O}, \{u_n\} \in J]$, $[\{u_n\}; \{g_n-u_n\} \in \tilde{O}, \{u_n\} \in J]$ and $[\{u_n\}; \{h_n-u_n\} \in \tilde{O}, \{u_n\} \in J]$ respectively.*

(i) $\{f_n\} \in \tilde{\mathfrak{f}}$, (ii) *If* $\{g_n\} \in \tilde{\mathfrak{f}}$, $\{f_n\} \in \tilde{\mathfrak{g}}$. (iii) *If* $\{g_n\} \in \tilde{\mathfrak{f}}$ *and* $\{h_n\} \in \tilde{\mathfrak{g}}$ *hold*, $\{h_n\} \in \tilde{\mathfrak{f}}$.

Proof. (i) holds evidently. (ii) If $\{f_n-g_n\} \in \tilde{O}$, $\{g_n-f_n\} \in \tilde{O}$ holds. Then (ii) holds. (iii) If $\{f_n-g_n\} \in \tilde{O}$ and $\{g_n-h_n\} \in \tilde{O}$, $\{f_n-h_n\} \in \tilde{O}$ holds from Lemma I-3. Then (iii) holds.

Classify J by $\tilde{O}$ and construct the space of the classes $\tilde{J}$. Namely the class $\tilde{\mathfrak{f}}$ (or $\mathfrak{f}$) corresponding to $\{f_n\} \in J$ is $[\{g_n\}; \{f_n-g_n\} \in \tilde{O}, \{g_n\} \in J]$. $\mathfrak{f}$ denotes the class $\tilde{\mathfrak{f}}$ regarded as the point in $\tilde{J}$.

Definition I-1. Let L_w denote the space $\Big[\mathfrak{f}$ (the equivalent class of $\{f_n\}$); $\lim_{n\to\infty}\int f_n^2\varphi dx=\int f^2\varphi dx$ for $\forall\varphi \in B$, where $f, f_n \in L^2_{(-\infty,\infty)}\Big]$.

Let L_d denote the space $[\mathfrak{f}$ (the equivalent class of $\{f_n\}$); $f_n^2 = f^2$, $f \in L^2_{(-\infty,\infty)}]$.

Let L_2 denote the linear space $[\mathfrak{f}$ (the equivalent class of $\{f_n\}$); $f_n = f \in L^2_{(-\infty,\infty)}]$ corresponding to $L^2_{(-\infty,\infty)}$ set-theoretically.

L_w, L_d, $L_2 \subseteq \tilde{J}$ holds. Let $f(x) \in L^2_{(-\infty,\infty)}$ satisfying $\|f(x)\|_{L^2} \neq 0$.

Since the equivalent class $\mathfrak{g}$ of $\{g_n\} \equiv \{f, -f, f, \cdots\}$ is contained in $L_d \cap L_2^c$, $L_w \supseteq L_d \supset L_2$ holds.

Let $\{f_n\}$ be $\{f, f, \cdots\}$. $\{f_n\}$ and $\{g_n\}$ are contained in J. But, since $\{f_n\}+\{g_n\}=\{2f(x), 0, 2f(x), 0, \cdots\}$ holds, $\{f_n\}+\{g_n\}$ is not contained in J. Then J (consequently $\tilde{J}$) is not a linear space. But J and $\tilde{J}$ contain the various linear subspaces. For example, $\tilde{O} \subseteq J$ and $L_2 \subseteq \tilde{J}$ are linear subspaces in J and $\tilde{J}$. If $\lim_{n\to\infty}\int f_n \cdot g_n \cdot \varphi dx$ for given two $\{f_n\}$, $\{g_n\} \in J$ becomes finite and definite for any $\varphi \in B$ (other than the inequality in the result of Lemma I-2), $\{f_n\} \pm \{g_n\} \in J$ holds.

Lemma I-5. *Let $\{f_n\}$, $\{g_n\}$ be the elements in J, and let $\{h_n^{(1)}\}$, $\{h_n^{(2)}\}$ be the elements in $\tilde{O}$. If a pair $\{f_n\}$, $\{g_n\}(\in J)$ has a definite and finite limit $\lim_{n\to\infty}\int f_n \cdot g_n \cdot \varphi dx$ for any $\varphi \in B$, $\lim_{n\to\infty}\int (f_n+h_n^{(1)})\cdot(g_n+h_n^{(2)})\cdot\varphi dx = \lim_{n\to\infty}\int f_n \cdot g_n \varphi dx$ holds for any $\varphi \in B$.*

Proof.

$$\left|\int f_n \cdot h_n^{(2)} \cdot \varphi dx\right| \leqslant \sqrt{\int f_n^2 dx \cdot \int h_n^{(2)2} \cdot \varphi^2 dx}, \left|\int h_n^{(1)} \cdot g_n \cdot \varphi dx\right| \leqslant \left|\sqrt{\int h_n^{(1)2} \cdot \varphi^2 dx \cdot \int g_n^2 dx}\right.$$

and $\left|\int h_n^{(1)} \cdot h_n^{(2)} \cdot \varphi dx\right| \leqslant \sqrt{\int h_n^{(1)2} dx \cdot \int h_n^{(2)2}\varphi^2 dx}$ hold for any $\varphi \in B$ from Schwarz inequality. Since φ^2 is also an element in B for any $\varphi \in B$ (Lemma I-1), $\lim_{n\to\infty}\int f_n \cdot h_n^{(2)} \cdot \varphi dx = \lim_{n\to\infty}\int h_n^{(1)} \cdot g_n \cdot \varphi dx = \lim_{n\to\infty}\int h_n^{(1)} \cdot h_n^{(2)} \cdot \varphi dx$

$=0$ holds for any $\varphi \in B$, and $\lim_{n\to\infty}\int (f_n+h_n^{(1)})\cdot(g_n+h_n^{(2)})\cdot\varphi dx$ $=\lim_{n\to\infty}\left\{\int f_n\cdot g_n\cdot\varphi dx+\int h_n^{(1)}\cdot g_n\cdot\varphi dx+\int f_n\cdot h_n^{(2)}\cdot\varphi dx+\int h_n^{(1)}\cdot h_n^{(2)}\cdot\varphi dx\right\}=\lim_{n\to\infty}$ $\int f_n\cdot g_n\cdot\varphi dx$ holds for any $\varphi \in B$.

Corollary. $\lim_{n\to\infty}\int (f_n+h_n^{(1)})^2\varphi dx=\lim_{n\to\infty}\int f_n^2\varphi dx$ *holds for any* $\varphi \in B$ *and for any given* $\{f_n\}\in J$ *and* $\{h_n^{(1)}\}\in \tilde{O}$.

Furthermore, it follows from this Lemma I-5 that the inner product $\langle \mathfrak{f}, \mathfrak{g}\rangle$ of two elements $\mathfrak{f}, \mathfrak{g}\in \tilde{J}$ equivalent to $\{f_n\}$, $\{g_n\}\in J$ respectively $\left(\text{with the definite and finite } \lim_{n\to\infty}\int f_n\cdot g_n\cdot\varphi dx \text{ for any } \varphi\in B\right)$ can be defined by $\lim_{n\to\infty}\int f_n\cdot g_n\cdot 1dx$ for any given $\{f_n\}\in\tilde{\mathfrak{f}}$ and any given $\{g_n\}\in\tilde{\mathfrak{g}}$. Because it determines unique limit (if it exists) independently of the choice of two elements $\{f_n\}$ and $\{g_n\}$ contained in $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ respectively. The orthonormal sequences in $\tilde{J}$ by $\langle \mathfrak{f}, \mathfrak{g}\rangle$ can be also defined.

§ 3. $\wedge$ ideal not to be a limit space. Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be two filters contained in τx ($x\in E$) relating to a limit space (E, τ).

Lemma I-6. $\mathfrak{F}_1\cap\mathfrak{F}_2$ *consists of the elements in* $(\mathfrak{F}_1)\cap(\mathfrak{F}_2)$.

Proof. If K is an element of $(\mathfrak{F}_1\cap\mathfrak{F}_2)$, $K\equiv F\cup G$ holds by $F\in(\mathfrak{F}_1)$ and $G\in(\mathfrak{F}_2)$. Since $F\cup G\supseteq F$ and $F\cup G\supseteq G$ hold, $F\cup G\in(\mathfrak{F}_1)\cap(\mathfrak{F}_2)$ holds from (F_1) in the filter's definition [1] p. 32. Namely $K\in(\mathfrak{F}_1)\cap(\mathfrak{F}_2)$. If $K\in(\mathfrak{F}_1)\cap(\mathfrak{F}_2)$, $K\in(\mathfrak{F}_1)$ and $K\in(\mathfrak{F}_2)$. Since $K=K\cup K$, K is the element of $(\mathfrak{F}_1\cap\mathfrak{F}_2)$.

Lemma I-7. *Let* $\tau x\equiv\{\mathfrak{F}\,;\,\mathfrak{F}\geqslant\mathfrak{F}_0(x)\}$ *be the set of filters constructed from a fixed filter* $\mathfrak{F}_0(x)$. *If any element of* $\mathfrak{F}_0(x)$ *contains* $x\in E$, τx *satisfies* (L^1) (L^2) *and* (L^3) *shown in* § 1, 1.2.

Proof. If $\mathfrak{F}_1\geqslant\mathfrak{F}_0(x)$ and $\mathfrak{F}_2\geqslant\mathfrak{F}_0(x)$ hold (i.e. $(\mathfrak{F}_1)$, $(\mathfrak{F}_2)\supseteq(\mathfrak{F}_0(x))$), $\mathfrak{F}_1\cap\mathfrak{F}_2\supseteq\mathfrak{F}_0(x)$ also holds from Lemma I-6. Then τx satisfies the condition of limit space (L^1) (i). Since $\bar{\mathfrak{F}}$ satisfying $\bar{\mathfrak{F}}\geqslant\mathfrak{F}$ for a given $\mathfrak{F}\in\tau x$ is contained in τx (from τx's definition), (L^1) (ii) evidently holds.

Since any element of $\mathfrak{F}_0(x)$ contains x, $[x]\geqslant\mathfrak{F}_0(x)$ also holds. Then τx satisfies (L^2). Since $\mathfrak{F}_0(x)$ is the weakest filter in τx, τx also satisfies (L^3).

Example I-1. Let (E, τ) be a limit space such that there exists a filter $\mathfrak{F}\in\tau x$ not equal to $[x]$. Let $\{A_\alpha-x\}$ be the family of (nonvoid) sets constructed from a filter $\mathfrak{F}=\{A_\alpha\}\in\tau x$ in (E, τ) not equal to $[x]$. Since $(A_\alpha-x)\cap(A_\beta-x)=(A_\alpha\cap A_\beta-x)\in\{A_\alpha-x\}$ holds from A_α, $A_\beta\in(\mathfrak{F})$, $\{A_\alpha-x\}$ becomes the base of a filter. Let $\mathfrak{F}^{(-x)}$ be the filter with the base $\{A_\alpha-x\}$, and $\tilde{\tau}x$ be the set of filters $\{\tilde{\mathfrak{F}}\,;\,\tilde{\mathfrak{F}}\geqslant\mathfrak{F}^{(-x)}\}$.

Theorem I-1. *The above space* $(E, \tilde{\tau})$ *satisfies* (L^1), *but it does not satisfy* (L^2).

Proof. (i) Let $\tilde{\mathfrak{F}}^{(1)}$ and $\tilde{\mathfrak{F}}^{(2)}$ be two filters finer than the one with the base $\{A_\alpha - x\}$, where $\mathfrak{F}=\{A_\alpha\} \in \tau x$ (not equal to $[x]$). Since $\tilde{\mathfrak{F}}^{(1)} \cap \tilde{\mathfrak{F}}^{(2)}$ is also finer than the one with the base $\{A_\alpha - x\}$, $\tilde{\mathfrak{F}}^{(1)} \cap \tilde{\mathfrak{F}}^{(2)} \in \tilde{\tau} x$ holds.

(ii) If $\bar{\mathfrak{F}}$ is the filter satisfying $\tilde{\mathfrak{F}} \leqslant \bar{\mathfrak{F}}$ by $\tilde{\mathfrak{F}} \in \tilde{\tau} x$, $\bar{\mathfrak{F}} \in \tilde{\tau} x$ holds, for $\bar{\mathfrak{F}} \geqslant \tilde{\mathfrak{F}} \geqslant \mathfrak{F}^{(-x)}$ holds.
Here $\mathfrak{F}^{(-x)}$ is the filter with the base $\{A_\alpha - x\}$ by $\mathfrak{F}=\{A_\alpha\} \in \tau x$ ($\mathfrak{F} \neq [x]$). Then $(E, \tilde{\tau})$ satisfies $(\boldsymbol{L}^1)$ from the above (i) (ii). Since $([x]) \not\supseteq (\mathfrak{F}^{(-x)})$ (i.e. $[x] \not\geqq \mathfrak{F}^{(-x)}$), $[x] \notin \tilde{\tau} x$, and $(E, \tilde{\tau})$ does not satisfy $(\boldsymbol{L}^2)$.

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