# 86. *Periodic Solutions of the Third Sorte for the Restricted Problem of Three Bodies*

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(Comm. by Yusuke HAGIHARA, M. J. A., May 12, 1969)

**Abstract.** Five periodic solutions with moderate eccentricities and high inclinations for the three-dimensional restricted problem of three bodies are found for cases of 3:2, 2:1, and 4:1 of the mean motions by expanding the disturbing functions by use of a high-speed computer. The expansion with respect to the inclination is made by Tisserand's polynomials and that to the eccentricity is made by Newcomb operators up to the tenth power. The periodic solutions found here correspond to orbits, for which secular and long-periodic perturbations of orbital elements except for the mean anomaly vanish. The existence of such periodic orbits are verified by numerical integration method for a case that the disturbing mass is 0.001.

**1. Introduction.** Equations of motion for three-dimensional restricted problem of three bodies, that is, for an asteroid moving under gravitational attractions of the Sun and Jupiter on a circular orbit, are written in canonical form by use of the following Delaunay variables:

$$\left.\begin{array}{ll} L=k\sqrt{a}, & l\text{: the mean anomaly,} \\ G=L\sqrt{1-e^2}, & g\text{: the argument of perihelion,} \\ H=G\cos i, & h\text{: the longitude of the ascending node,} \\ & \quad \Omega\text{, minus the longitude of Jupiter,} \end{array}\right\} \quad (1)$$

where $a$, $e$, and $i$ are, respectively, the semi-major axis, the eccentricity, and the orbital inclination to Jupiter's orbital plane for the asteroid, and $k$ is the gravitational constant of Gauss. The units are so chosen that the mass of the Sun and the mean motion and the semi-major axis of Jupiter are unity.

Short-periodic terms which depend on $l$ and/or $h$ can be eliminated from the Hamiltonian by von Zeipel's transformation, for example, and, therefore, the equations of motion are reduced to those of one degree of freedom since $L$ and $H$ are constant after the transformation. Then the values of $G$ can be derived as a function of $g$ by solving the energy integral, and if the inclination is sufficiently high, stationary solutions, in which $G$ and $g$ are constant, are found for $2g=180°$.[1)]

However, if the mean motion, $n$, of the asteroined is nearly commensurable with Jupiter's mean motion which is equal to 1, the degree of freedom of the equations of motion can be reduced to two but not to one by eliminating short-periodic terms since $l$ and $h$ are not independent for this case. For a commensurable case, in which the mean motion is nearly equal to $(p+q)/q$ with two integers $p$ and $q$, the equations of motion are conveniently expressed by the following canonical variables;

$$\left.\begin{aligned} X_1&=[(p+q)L-pH]/q, & Y_1&=\lambda-t,\\ X_2&=(L-H)/q, & Y_2&=(p+q)t-p\lambda-q\tilde{\omega},\\ X_3&=G-H, & Y_3&=g, \end{aligned}\right\} \tag{2}$$

where $\lambda$, $t$, and $\tilde{\omega}$ are, respectively, the mean longitudes of the asteroid and Jupiter and the longitude of the perihelion. The Hamiltonian $F$ for these canonical variables is written as,

$$F=k^4(X_1-pX_2)^{-2}/2+X_1-(p+q)X_2+m'k^2R, \tag{3}$$

where $m'$ is the mass of Jupiter and $R$ is the disturbing function.

Short-periodic terms which depend on $Y_1$ in $R$ can be eliminated from the Hamiltonian by von Zeipel's transformation which reduces the equations to those of two degrees of freedom. Although they can not be generally solved, stationary solutions, in which $X_i$ and $Y_i$ for $i=2$ and 3 are constant, can be found. In this paper such particular solutions are found by expanding the disturbing function by the high-speed computer, HITAC 5020E at the Computer Center, the University of Tokyo.

**2. Disturbing function.** The disturbing function $R$ due to Jupiter is written as

$$R=\Delta^{-1}-r\cos S, \tag{4}$$

where $\Delta=\sqrt{1+r^2-2r\cos S}$ is the linear distance and $S$ is the heliocentric angular distance between the asteroid and Jupiter while $r$ is the heliocentric distance of the asteroid. When the eccentricity is zero $\Delta^{-1}$ is expanded into a Fourier series with argument $S$, in which the true anomaly is replaced by the mean anomaly, as,

$$\Delta^{-1}=b_0+2\sum_{n=1}^{\infty} b_n \cos nS, \tag{5}$$

where $b_n$ is Laplace coefficient which is a function of $a$. Laplace coefficients and their Newcomb derivatives $D^j b_n$, where $D$ is the differential operator $a(d/da)$, can be computed by the method of Izsak and Benima[2] when $a$ is given.

The trigonometric function $\cos nS$ can be represented by a finite sum of trigonometric terms with linear combination of $\lambda$, $t$, and $2\Omega$ as argument and with polynomials of $\sin^2(i/2)$ as coefficient by use of Tisserand's polynomials.[3] Then $R$ for $e=0$ can be written as

$$R=b_{00}+2\sum_{\nu=1}^{\infty} b_{0\nu}\cos\nu\psi+2\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\infty} b_{\mu\nu}\cos(\mu\varphi-\nu\psi), \tag{6}$$

where $\varphi=\lambda-t$, $\psi=\lambda+t-2\Omega$, and $b_{\mu\nu}$ is a linear function of $b_n$ with Tisserand's polynomials as coefficients. The coefficients $b_{10}$ and $b_{01}$ take different forms because contributions from $r\cos S$ appear in these two coefficients. These expressions are valid for any inclination.

The eccentricity, $e$, is introduced by use of Newcomb operators which are polynomials of the differential operator $D$ with polynomials of $\mu-\nu$, the coefficient of $\lambda$ in (6), as coefficients. Newcomb operators to include the eccentricity up to the tenth power are computed by the method of Izsak *et al.*[4] After these computations are made, the disturbing function is expanded in the form,

$$R=\sum_{j=-10}^{10}\sum_{\mu}\sum_{\nu}C_{j\mu\nu}\cos\left[-jl+(\mu-\nu)\lambda-(\mu+\nu)t+2\nu\Omega\right], \qquad (7)$$

where $C_{j\mu\nu}$ is a function of $a$, $e$, and $i$ and contains $e^{|j|}\sin^{2|\nu|}(i/2)$ as a factor. Since the short-periodic terms can be eliminated, it is not necessary to compute terms in (7) unless $\mu$ and $\nu$ satisfy the following condition;

$$(p+q)(\mu-\nu-j)=p(\mu+\nu),\ \text{or,}\ q\mu=(2p+q)\nu+(p+q)j. \qquad (8)$$

The computations are made for 3/2, 5/3, 2, 3, 4, and 5, for which the upper limits of the eccentricity are, respectively, 0.18, 0.2, 0.3, 0.6, 0.62, and 0.62.

**3. Stationary solutions.** After the short-periodic terms depending on $Y_1$ are eliminated, the disturbing function is expressed as,

$$R=\sum_{j=-10}^{10}\sum_{\nu}B_{j\nu}\cos\left(\frac{2\nu+j}{q}Y_2+2\nu Y_3\right). \qquad (9)$$

Stationary solutions are obtained by solving the equations,

$$\frac{dX_i}{dt}=\frac{\partial F}{\partial Y_i}=0, \quad \frac{dY_i}{dt}=-\frac{\partial F}{\partial X_i}=0, \quad (i=2, 3). \qquad (10)$$

The last two equations are satisfied for $Y_2=0°$ or $180°$ and $2Y_3=0°$ or $180°$.

The expression of the third equation is

$$\frac{dY_2}{dt}=(p+q)-pk^4(X_1-pX_2)^{-3}-m'k^2\frac{\partial R}{\partial X_2}$$
$$=(p+q)-pn+m'k^2\left[p\frac{\partial R}{\partial L}+(p+q)\left(\frac{\partial R}{\partial G}+\frac{\partial R}{\partial H}\right)\right]=0, \qquad (11)$$

where the mean motion, $n$, is computed by $na^{3/2}=k$. When $e$ and $i$ are given, this equation is satisfied by changing the value of $a$ or $n$ by an amount of order of $m'$.

The last equation is

$$\frac{dY_3}{dt}=-m'k^2\frac{\partial R}{\partial X_3}=-m'k^2\frac{\partial R}{\partial G}=0. \qquad (12)$$

This equation has been solved by Jefferys and Standish[5] by a method of numerical integration. The equation can be also solved by using the expansion (9) of the disturbing function and the solutions are expressed as curves in $(e, i)$-plane when $Y_2$ and $Y_3$ are given. For each pair of $e$ and $i$ on these curves the value of $a$ is fixed so that the equation (11) is satisfied.

Thus the stationary solutions have been found by the semi-analytical method as far as secular and long-periodic perturbations are concerned.

4. **Periodic solutions.** However, if the short-periodic perturbations are taken into consideration, $X_i$ and $Y_i$ $(i=2, 3)$ are no more constant but oscillate about the stationary values. For the stationary solutions the longitude of the ascending node, and, therefore, the orbital plane is moving even if the short-periodic perturbations are not included. Therefore, the stationary solutions thus found for both commensurable and non-commensurable cases are not usually periodic.

However, if the mean value of $dY_1/dt$, which is written as,

$$\frac{dY_1}{dt} = -\frac{\partial F}{\partial X_1} = n-1-m'k^2\left(\frac{\partial R}{\partial L}+\frac{\partial R}{\partial G}+\frac{\partial R}{\partial H}\right), \tag{13}$$

is exactly equal to $q/p$, the short-periodic perturbations in all the orbital elements take the same values as the initial ones after $p$ revolutions of the asteroid. This condition is satisfied when the mean values of $\partial R/\partial G$ and $\partial R/\partial H$ vanish and that of $n-m'k^2(\partial R/\partial L)$ is equal to $q/p$. Under these conditions the equations (11) and (12) are satisfied and the longitude of the ascending node does not move if the short-periodic perturbations are not included. Thus periodic solutions of the third sorte studied by Poincaré[6] can be found by obtaining stationary solutions which satisfy the above conditions.

The solutions for $\partial R/\partial H=0$ are also expressed as curves in $(e, i)$-plane, and if two curves for $\partial R/\partial G=0$ and $\partial R/\partial H=0$ intersect there exists a periodic solution. However, it is found that such periodic solutions cannot be obtained for even values of $q$ in $n=(p+q)/p$, unless very eccentric orbits are considered. For $n=4$, 2, and 3/2 two curves intersect, and, therefore, periodic solutions of the third sorte have been found. The orbits corresponding to these periodic solutions have been computed for $m'=0.001$ by a method of numerical integrations, and osculating elements at the time of conjunction of Jupiter and the asteroid for each of the periodic solutions are derived. They are shown in Table I. Of these solutions the orbit IV is found by purely numerical way.

For $Y_2=0°$ the conjunction always takes place when the asteroid passes through the perihelion, and for $Y_2=180°$ it does when the asteroid is at the aphelion. For III and V the conjunctions take place at the aphelia, although the perihelia and aphelia are far from the orbital plane of Jupiter as the arguments of perihelion for these orbits

Table I. Osculating elements for conjunction

| | $n$ | $i$ | $e$ | $Y_2$ | $2Y_3$ |
|---|---|---|---|---|---|
| I | 4.000 136 | 90°.126 28 | 0.014 18889 | 0° | 180° |
| II | 3.999 005 | 88.674 31 | 0.113 9580 | 0 | 0 |
| III | 1.999 302 | 95.235 29 | 0.171 8233 | 180 | 180 |
| IV | 1.498 002 | 69.617 81 | 0.426 5092 | 0 | 0 |
| V | 1.497 947 | 98.104 10 | 0.189 7712 | 180 | 180 |

are 90°. The solution IV is very interesting since the aphelion of the orbit is outside of Jupiter's orbit as the aphelion distance is 1.09. However, since the conjunction takes place always at the perihelion, the asteroid never approaches very closely to Jupiter even for this case. For all the solutions given in Table I the asteroid and Jupiter cannot approach to each other very closely.

## References

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