81. *On Generalized Integrals.* V

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Many real variable integrals have been defined as extensions of the Lebesgue integral. Some of them depend heavily on special properties of the derivative on the real line, and some follow the idea of definite integral as limits of certain approximating sums. The idea of Kunugi's generalized integrals follows the latter direction. As it is already known (IV,[1] Theorem 7) [4], for the $(E.R.\varphi)$ integrable function $f(x)$, the integral is defined as limit of approximating sums $\int_{G_n} f(x)dx$, $G_n=\{x\,;\,f(x)\leq n\varphi(x)\}$. In this case, the sequence $\{G_n\}$ satisfies the condition $\lim_{n\to\infty} n\int_{G_n}\varphi(x)dx=0$. Theorem 9 shows that the Denjoy-integrable function $f(x)$ in the general (resp. special) sense in $[a,b]$ is a measurable function for which there exists a monotone increasing sequence $\{F_n\}$, with union $[a,b]$, of closed sets with the properties [C] and [D] (resp. $[D_*]$), and then the integral is given as limit of approximating sums $\int_{F_n} f(x)dx$. On the other hand, we have seen that for each positive Lebesgue-integrable function φ, the sets of $(E.R.\varphi)$ integrable functions and Denjoy-integrable functions in either the general or the special sense partially intersect. Moreover, for some functions which are both $(E.R.\varphi)$ and Denjoy-integrable, even where the Denjoy integrability is in the special sense, both integrals do not coincide.[2] Thus, we see that the $(E.R.)$ integrability differs essentially from the Denjoy integrability, and the difference between the methods of totalization of the $(E.R.)$ integral and the Denjoy

1) The reference number indicates the number of the Note.

2) This has been proved by I. A. Vinogradova for the case of special $(E.R.)$ integral (i. e. A-integral) (see [6]). On the other hand, if $\varphi(t)$ is a positive Lebesgue-integrable function in $[a,b]$, and if $\Phi(t)$ is the indefinite integral of $\varphi(t)$ such that $\Phi(a)=\alpha$ and $\Phi(b)=\beta$, then for a function $f(x)$ defined in $[\alpha,\beta]$, we see that: (1) $f(\Phi(t))\varphi(t)$ is $(E.R.\varphi)$ integrable in $[a,b]$ if and only if $f(x)$ is $(E.R.)$ integrable in the special sense in $[\alpha,\beta]$, and then $(E.R.\varphi)\int_a^b f(\Phi(t))\varphi(t)dt=(E.R.)\int_\alpha^\beta f(x)dx$ (see IV), and (2) $f(\Phi(t))\varphi(t)$ is Denjoy-integrable in the general (resp. special) sense in $[a,b]$ if and only if $f(x)$ is Denjoy-integrable in the general (resp. special) sense in $[\alpha,\beta]$, and then $(D)\int_a^b f(\Phi(t))\varphi(t)dt=(D)\int_\alpha^\beta f(x)dx$. Therefore, it follows that the assertion is also true for the general case.

integral appears in the form of the difference between the sequences of sets $\{G_n\}$ and $\{F_n\}$ reasonably chosen.

In this paper, we first give a few sufficient conditions that a function $f(x)$ in $[a, b]$ for which there exists the limit $\lim_{n\to\infty}\int_{F_n} f(x)dx$, where $\{F_n\}$ is a monotone increasing sequence of closed sets, become a function, $(E.R.)$ integrable with respect to φ reasonably chosen, such that the limit value $\lim_{n\to\infty}\int_{F_n} f(x)dx$ is taken as the value $(E.R.\varphi)\int_a^b f(x)dx$ of the integral (Propositions 17, 18). As applications, we get Theorems 8 and 10. K. Fujita has proved Theorem 10 in the form of $(E.R.\nu)$ integral in [1]. This Theorem shows that for a Denjoy-integrable function $f(x)$ in the general sense, if we choose φ reasonably, $f(x)$ is also $(E.R.\varphi)$ integrable and both integrals are given as limit of the same approximating sums.

We shall assume throughout this paper that the interval is finite. We denote by $[a, b]$ the closed interval, and denote by $[a, b)$ the half-open interval on the right side. We conserve the terminologies and the notations of the preceding papers I–IV [4].

8. The (E.R. φ) integrals and the Denjoy integrals. Let us begin with giving the following Lemma without proof.

Lemma 24. *If $f(x)$ is a function in $[a, b)$ which is bounded on any proper subinterval of $[a, b)$, then there exists a function $u(x)$ in $[a, b)$ with the following properties:*

1) $|f(x)| \leq u(x)$ *for all* $x \in [a, b)$,

2) $u(a) \geq 1$, $\lim_{x\to b} u(x) = +\infty$, *$u(x)$ is continuously differentiable and* $u'(x) \geq 1$ *at any point x of* $[a, b)$.

Lemma 25. *If $f(x)$ is a measurable function in $[a, b)$ for which there exists a $u(x)$ with the properties* 1) *and* 2) *of Lemma* 24, *such that* $\int_a^{b'} f(x)dx$ *converges to a finite limit when* $b' \to b$, *for $\varphi(x)$ defined by* $\varphi(x) = e^{u(x)-w(x)}$, $w(x) = e^{u(x)}$, *$f(x)$ is $(E.R.\varphi)$ integrable and* $(E.R.\varphi)\int_a^b f(x)\,dx = \lim_{b\to b'}\int_a^{b'} f(x)dx$ *holds.*

Proof. It is easy to verify that $\varphi(x)$ is a positive Lebesgue-integrable function in $[a, b)$. For the sequence $\{\varepsilon_n\}$, $\varepsilon_n = 2^{-n}$, let $\{b_n\}$ be a monotone increasing sequence converging to b, such that $\left|\int_{b_n}^x f(x)dx\right| < \varepsilon_n/3$ for every x with $b_n < x < b$, $u(b_n)e^{-u(b_n)} < \varepsilon_n/3$ and $\int_{b_n}^b \varphi(x)dx < \varepsilon_n/3$. Put $F_n = [a, b_n]$, and put $f_n(x) = f(x)$ on F_n and zero elsewhere. Then, $\{f_n\}$ is a sequence of functions r-converging to f in $\{\mathcal{M}(a, b), \varphi\}$.[3] In fact, we see that $\{V(F_n, \varepsilon_n; f)\}$ is a fundamental

3) For the definition, see IV.

sequence with $f_n \in V(F_n, \varepsilon_n; f)$. It is easy to verify that the sequence is fundamental. Moreover, $[\alpha(\varphi)]$ $|f_n(x)-f(x)|=0$ for all $x \in F_n$. $[\beta(\varphi)]$ Since $u(x)e^{-u(x)+w(x)}$ is a strictly monotone increasing continuous function, for every sufficiently great value of k, there exists one and only one point $x_0=x_0(k)$ such that $k=u(x_0)e^{-u(x_0)+w(x_0)}$. Hence, if we put $E_k=\{x;\ |f(x)|>k\varphi(x)\}$, we have $E_k \subseteq [x_0, b)$. Therefore, $k\int_{\{x;|f_n(x)-f(x)|>k\varphi(x)\}}\varphi(x)dx \leq k\int_{\max(x_0,b_n)}^{b}\varphi(x)dx = ke^{-w(\max(x_0,b_n))}$. Since, furthermore, $u(x)e^{-u(x)}$ and $u(x)e^{-u(x)+w(x)}$ are respectively monotone decreasing and increasing functions, we have $ke^{-w(\max(x_0,b_n))} \leq u(b_n)e^{-u(b_n)}$, and so $k\int_{\{x;|f_n(x)-f(x)|>k\varphi(x)\}}\varphi(x)dx<\varepsilon_n/3<\varepsilon_n$. $[\gamma(\varphi)]$ First, in the case when $x_0(=x_0(k))\geq b_n$, we have $\left|\int_a^b [f(x)-f_n(x)]^{k\varphi(x)}dx\right| \leq \left|\int_{b_n}^{x_0} f(x)dx\right| + k\int_{[x_0,b)\cap E_k}\varphi(x)dx + \left|\int_{[x_0,b)\setminus E_k} f(x)dx\right| < 2\varepsilon_n/3 + \int_{x_0}^{b} ke^{u(x)-w(x)}dx < \varepsilon_n$. In the case when $x_0<b_n$, we have $\left|\int_a^b [f(x)-f_n(x)]^{k\varphi(x)}dx\right| \leq k\int_{[b_n,b)\cap E_k}\varphi(x)dx + \left|\int_{[b_n,b)\setminus E_k} f(x)dx\right| < \varepsilon_n/3 + \int_{b_n}^{b} ke^{u(x)-w(x)}dx < \varepsilon_n$. Thus, $f_n \in V(F_n, \varepsilon_n; f)$. Consequently, from the fact that the $(E.R.\varphi)$ integral is the r-continuous extension of the Lebesgue integral in $\{\mathscr{M}(a,b), \varphi\}$ (see IV), it follows that $f(x)$ is $(E.R.\varphi)$ integrable and $\lim_{n\to\infty}\int_a^b f_n(x)dx = (E.R.\varphi)\int_a^b f(x)dx$. Moreover, from the assumption, $\lim_{b'\to b}\int_a^{b'} f(x)dx = (E.R.\varphi)\int_a^b f(x)dx$ follows.

Lemma 26. *Let $\{F_n\}$ be a monotone increasing sequence, with union $[a,b]$ except for a set of measure zero, of closed sets such that $F_{n+1}\setminus F_n$ is a closed set, and let $\tau(x)$ be the function defined by*

$$\tau(x) = \mathrm{mes}([a,x]\cap(F_{n+1}\setminus F_n)) + \mathrm{mes}\, F_n, \text{ if } x \in F_{n+1}\setminus F_n,$$
$$(n=0,1,2,\cdots;\ F_0=\phi).$$

Then, 1) *$\tau(x)$ is a mapping of $\bigcup_n F_n$ onto $[0, b-a)$, which is a one to one mapping except for a set N of measure zero such that the range $\tau(N)$ is a set of countable points,* 2) *if $f(y)$ is a Lebesgue-integrable function in $[0, b-a)$, then we have $\int_{\tau(E)} f(y)dy = \int_E f(\tau(x))dx$ for every measurable set E in $[a,b]$.*

We now obtain the following proposition:

Proposition 17. *Let $f(x)$ be a measurable function in $[a,b]$ with the following property: there exists a monotone increasing sequence $\{F_n\}$, with union $[a,b]$ except for a set of measure zero, of closed sets in $[a,b]$ on each of which $f(x)$ is Lebesgue-integrable, such that:* $[C]$[4)]

4) Cf. [3], condition $[3_1]$.

for every interval I, $\int_{F_n\cap I} f(x)dx$ converges to a finite limit when $n\to\infty$, and the convergence is uniform with respect to I. Then, there is a positive Lebesgue-integrable function $\varphi(x)$ in $[a,b]$ such that $f(x)$ is $(E.R.\varphi)$ integrable in any interval I in $[a,b]$ and $(E.R.\varphi)\int_I f(x)dx$ $=\lim_{n\to\infty}\int_{F_n\cap I} f(x)dx$.

Proof. Let us consider a monotone increasing sequence $\{A_n\}$ of closed sets such that $A_n\subseteq F_n$, $A_{n+1}\setminus A_n$ is a closed set, $\lim_{n\to\infty}\text{mes}\,A_n=b-a$, $\int_{F_n\setminus A_n}|f(x)|dx<2^{-(n+1)}$ and $f(x)$ is bounded on every A_n. Let $\tau(x)$ be the mapping of $\bigcup_n A^n$ onto $[0,b-a)$ defined in Lemma 26 for $\{A_n\}$. Then, $\tau(x)$ is a one to one mapping except for a set N of measure zero with $\text{mes}\,N^*=0$, $N^*=\tau(N)$. Put $f^*(y)=f(\tau^{-1}(y))$ on $[0,b-a)\setminus N^*$ and zero elsewhere. Let $u(y)$ be a function with the properties 1) and 2) of Lemma 24 for $f^*(y)$ and put $\psi(y)=e^{u(y)-w(y)}$, $w(y)=e^{u(y)}$. Then, since $\left|\int_{A_n\cap I}f(x)dx-\int_{F_n\cap I}f(x)dx\right|\le\int_{F_n\setminus A_n}|f(x)|dx<2^{-(n+1)}$ for every I, $\lim_{n\to\infty}\int_{A_n\cap I}f(x)dx$ exists and the convergence is uniform with respect to I. We now put $f_I^*(y)=f(\tau^{-1}(y))$ on $\tau((\bigcup_n A_n)\cap I)\setminus N^*$ and zero elsewhere, and put $\alpha_n=\text{mes}\,A_n$. Then, since, for b' with $0<b'<b-a$, there is $c_{n'}$ such that $c_{n'}\in A_{n'}\setminus A_{n'-1}$ and $\tau(c_{n'})=b'$, we have, for every I, $\int_a^{b'}f_I^*(y)dy=\int_{A_{n'-1}\cap I}f(x)dx+\int_{[a,c_{n'}]\cap(A_{n'}\setminus A_{n'-1})\cap I}f(x)dx$, so that there exists $\lim_{b'\to b-a}\int_a^{b'}f_I^*(y)dy$ and the convergence is uniform with respect to I. Consequently, there exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that, whenever $n\ge n_i$, $\left|\int_{\alpha_n}^{y}f_I^*(y)dy\right|<2^{-(i+4)}$ for every y with $\alpha_n<y<b-a$ and for every interval I, and that $u(\alpha_{n_i})e^{-u(\alpha_{n_i})}<2^{-(i+4)}$ and $\int_{\alpha_{n_i}}^{b-a}\psi(y)dy$ $<2^{-(i+4)}$. Hence, as it is seen in the proof of Lemma 25, if we put $\varepsilon_n=2^{-i}$ for all n such that $n_i\le n<n_{i+1}$, $\{V([0,\alpha_n],\varepsilon_n/4;f_I^*)\}$ is a fundamental sequence in $\{\mathcal{M}(0,b-a),\psi\}$ with $g_{I,n}^*\in V([0,\alpha_n],\varepsilon_n/4;f_I^*)$, where $g_{I,n}^*(y)=f^*(y)$ on $\tau(A_n\cap I)\setminus N^*$ and zero elsewhere. Consider the functions $f_I, f_{I,n}$ and $g_{I,n}$ defined in such a way that, $f_I(x)$ is the restriction on I of $f(x)$, $f_{I,n}(x)=f(x)$ on $F_n\cap I$ and zero elsewhere, and $g_{I,n}(x)$ $=f(x)$ on $A_n\cap I$ and zero elsewhere. Then, the sequence $\{V(F_n\cap I,\varepsilon_n;f_I)\}$ is fundamental in $\{\mathcal{M}(I),\psi(\tau)\}$. Moreover, we have $f_{I,n}\in V(F_n\cap I,\varepsilon_n;f_I)$. For, $[\alpha(\psi(\tau))]\,|f_{I,n}(x)-f_I(x)|=0$ for all $x\in F_n\cap I$, $[\beta(\psi(\tau))]\;k\int_{\{x;|f_{I,n}(x)-f_I(x)|>k\psi(\tau(x))\}}\psi(\tau(x))dx\le k\int_{\{y;|g_{I,n}^*(y)-f_I^*(y)|>k\psi(y)\}}\psi(y)dy$

$<\varepsilon_n/4$. $[\gamma(\psi(\tau))]\left|\int_a^b[f_{I,n}(x)-f_I(x)]^{k\psi(\tau(x))}dx\right|=\left|\int_a^b[g_{I,n}(x)-f_I(x)]^{k\psi(\tau(x))}dx\right.$ $\left.-\int_a^b[g_{I,n}(x)-f_{I,n}(x)]^{k\psi(\tau(x))}dx\right|\leq\left|\int_a^b[g_{I,n}(x)-f_I(x)]^{k\psi(\tau(x))}dx\right|$ $+k\int_{\{x;|g_{I,n}(x)-f_I(x)|>k\psi(\tau(x))\}}\psi(\tau(x))dx+\int_{F_n\setminus A_n}|f(x)|dx<\varepsilon_n$. Thus, $f_I\in\{\lim_n f_{I,n}\}$ in $\{\mathcal{M}(I),\psi(\tau)\}$. Consequently, $\varphi=\psi(\tau)$ is a desired function.

This proof shows that:

Remark 1. In Proposition 17, we can choose φ in such a way that, when we put $f_n(x)=f(x)$ on F_n and zero elsewhere, for every interval I, the sequence $\{f_n\}$ does not only r-converge to f in $\{\mathcal{M}(I),\varphi\}$, but satisfies the condition: [1] for every $k>0$, the convergence of $\int_I[f(x)-f_n(x)]^{k\varphi(x)}dx$ is uniform with respect to I.

Paying attention to the proof of Proposition 17, we get:

Proposition 18. *If we replace in Proposition* 17 *the condition* [C] *by the following condition*: [C_*] $\int_{F_n}f(x)dx$ *converges to a finite limit when* $n\to\infty$, *and* $\lim_{n\to\infty}\int_{(F_{n+1}\setminus F_n)\cap I}f(x)dx=0$ *for every interval* I *in* $[a,b]$ *and the convergence is uniform with respect to* I. *Then, there is a positive Lebesgue-integrable function* $\varphi(x)$ *in* $[a,b]$ *such that* $f(x)$ *is* ($E.R.\varphi$) *integrable in* $[a,b]$ *and* $(E.R.\varphi)\int_a^b f(x)dx=\lim_{n\to\infty}\int_{F_n}f(x)dx$.

We get the following theorem as an application of Proposition 18.

Theorem 8. *If* $f(x)$ *is a measurable function in* $[a,b]$ *such that* $\int_a^b f^+(x)dx=+\infty$ *and* $\int_a^b f^-(x)dx=-\infty$. *Then, for any real number* κ, *there exists a Lebesgue-integrable function* $\varphi(x)$ *for which* $f(x)$ *is* ($E.R.\varphi$) *integrable and* $(E.R.\varphi)\int_a^b f(x)dx=\kappa$.

Proof. For the case when $\kappa>0$, let $\{a_n\}$ be a monotone decreasing sequence of positive numbers such that $\sum a_n=\kappa$. For each n, let i_n be the smallest positive integer i such that $ia_n>1$. Put, for $m=1,2,\cdots$, $b_j=a_m$ if $j=\sum_{n=1}^{m-1}2i_n+2k-1$ $(k=1,2,\cdots,i_m-1)$, $b_j=-a_m$ if $j=\sum_{n=1}^{m-1}2i_n+2k$ $(k=1,2,\cdots,i_m)$ and $b_j=2a_m$ if $j=2i_m-1$. Then, $\sum b_j=\kappa$. By the assumption, there is a monotone increasing sequence $\{A_j\}$ (resp. $\{B_j\}$) of closed sets such that $\int_{A_j\setminus A_{j-1}}f^+(x)dx=b_{2j-1}$ $\left(\text{resp. }\int_{B_j\setminus B_{j-1}}f^-(x)dx=b_{2j}\right)$ $(A_0=\phi$ (resp. $B_0=\phi$)) and $\lim_{n\to\infty}\text{mes}\,A_j=\text{mes}\,\{x;f(x)\geq 0\}$ (resp. $\lim_{j\to\infty}\text{mes}\,B_j=\text{mes}\,\{x;f(x)<0\}$). If we put $F_j=A_j\cup B_j$, the desired assertion follows from Proposition 18. The proof is similar for the case $\kappa \leq 0$.

Theorem 9 (*Characterizations of Denjoy integrals*). *A necessary and sufficient condition that a measurable function $f(x)$ in $[a,b]$ be Denjoy-integrable in the general (resp. special) sense is that there exists a monotone increasing sequence $\{F_n\}$, with union $[a,b]$, of closed sets with the property* [C], *such that*: [D] (*resp.* [D_*])[5] *for each $\varepsilon>0$, there is an $n=n(\varepsilon)$ such that $n(\varepsilon)\to+\infty$ when $\varepsilon\to 0$, and whenever $n'\geq n$, $\sum_i \left|\int_{(F_{n'}\setminus F_n)\cap I_i} f(x)dx\right|<\varepsilon$ for every finite sequence $\{I_i\}$ of non-overlapping intervals whose end-points (resp. at least one of the end-points) belong to F_n. Then, we have $(D)\int_I f(x)dx$ $\left(\text{resp. } (D_*)\int_I f(x)dx\right)$ $=\lim_{n\to\infty}\int_{F_n\cap I} f(x)dx$ for every interval I in $[a,b]$.*

Proof. *Necessity.* The case of "special" follows from [2], Theorem 1. The case of "general" can be proved in the same way as in [1], lemma. *Sufficiency.* The case of "special" follows from [2], Theorem 5. For the case of "general", if we put $F(x)=\lim_{n\to\infty}\int_{F_n\cap[a,x]} f(x)dx$, then by [C], $F(x)$ is continuous. Moreover, we see that, in the same way as in [3], Theorems 5 and 7, $F(x)$ is absolutely continuous in the general sense on every F_n and $F'_{ap}(x)=f(x)$ a.e. Hence, $f(x)$ is Denjoy-integrable in the general sense (see [5]).

From Proposition 17 and Theorem 9, it follows that:

Theorem 10. *If $f(x)$ is a general (resp. special) Denjoy-integrable function in $[a,b]$, then there is a positive Lebesgue-integrable function $\varphi(x)$ in $[a,b]$ such that $f(x)$ is $(E.R.\varphi)$ integrable in every interval I in $[a,b]$ and $(D)\int_I f(x)dx$ $\left(\text{resp. } (D_*)\int_I f(x)dx\right)=(E.R.\varphi)$ $\int_I f(x)dx$.*

References

[1] K. Fujita: On indefinite ($E.R.$)-integrals. I, II. Proc. Japan Acad., **41**, 686-695 (1965).

[2] S. Nakanishi (formerly S. Enomoto): Sur une totalisation dans les espaces de plusieurs dimensions. I. Osaka Math. Jour., **7**, 69-102 (1955).

[3] ——: Sur la dérivation de l'intégrale ($E.R$) indéfinie. I. Proc. Japan Acad., **34**, 199-204 (1958).

[4] ——: On generalized integrals. I-IV. Proc. Japan Acad., **44**, 133-138, 225-230, 904-909 (1968), **45**, 86-91 (1969).

[5] S. Saks: Theory of the Integral. Warsaw (1937).

[6] I. A. Vinogradova: On indefinite A-integrals. Izv. Akad. Nauk SSSR Ser. Mat., **25**, 113-142 (1961).

5) The condition [D_*] includes the condition [C].