77. On the Structure of Certain C*-Algebras

By Tadashi HURUYA

(Comm. by Kunihiko KODAIRA, M. J. A., May 12, 1969)

A C*-algebra A is said to be *elementary* if A is isomorphic to the C*-algebra $LC(H)$ of the totality of compact operators on a Hilbert space H. The dual $\hat{A}$ of any elementary C*-algebra A consists of a single element (cf. 4.1.5 in [1]), conversely any separable C*-algebra is elementary if its dual consists of a single element [5], where the dual $\hat{A}$ is the set of all unitary equivalence classes of irreducible representations* of A. A C*-algebra A is called a CCR-algebra if $\pi(A)\subset LC(H_\pi)$ and a GCR-algebra if $\pi(A)\cap LC(H_\pi)\neq(0)$ (cf. [6]), for all irreducible representations π of A, where H_π denotes the representation space of π.

In this paper, we present some results on the structure of GCR-algebras whose dual consists of a finite number of elements.

Lemma 1. *If A is a separable* C*-*algebra and Card* $\hat{A}\leqq\aleph_0$, *then A is of type I.*

Proof. Let π be an irreducible representation of A. If $\pi(A)\cap LC(H_\pi)=(0)$, then, by [2], there is a family of mutually inequivalent irreducible representations of A which has the cardinal number of continuum. This fact is contrary to our assumption. Therefore we have $\pi(A)\cap LC(H_\pi)\neq(0)$ and, by [4] and [6], A is a C*-algebra of type I.

Lemma 2. *Let $(A_i)_{i\in I}$ be a family of non-zero* C*-*algebras and let A be the product* C*-*algebra of A_i's. Then we have*

$$\hat{A}=\bigcup_{i\in I}\{\rho_\pi\,|\,\pi\in\hat{A}_i\} \tag{1}$$

if and only if the index set I is a finite set, where ρ_π is a representation $(x_i)_{i\in I}\to\rho_\pi((x_i)_{i\in I})=\pi(x_i)$ of A.

Proof. Suppose that (1) is satisfied. Let B be the restricted product C*-algebra of A_i's (cf. 1.9.14 in [1]). Then B is a closed two-sided ideal of A. Assume that $B\subsetneqq A$, then there is an irreducible representation ν such that $\nu(B)=(0)$. By the assumption, there is an irreducible representation π of A_i, such that $\nu=\rho_\pi$. Then we have $(0)=\nu(B)=\rho_\pi(B)=\pi(A_i)=\rho_\pi(A)=\nu(A)$. This is a contradiction. Therefore $B=A$. Since A_i is non zero, I is a finite set.

Conversely, let I be a finite set. Each A_i is a closed two-sided ideal of A. Let π be an irreducible representation of A. Since I is a finite set, there is an index $i\in I$ such that $\pi(A_i)\neq(0)$. Let $j\in I$ be any

*) Throughout this paper, we mean by an irreducible representation a non-trivial one.

index other than i, and let x_i and x_j be any elements of A_i and A_j, respectively. Then we have

$$(\pi\,|\,A_i)(x_i)\pi(x_j)=\pi(x_ix_j)=\pi(0)=0.$$

Since $\pi\,|\,A_i$ is an irreducible representation of A_i and $H_{\pi|A_i}$ is H_π, we have $\pi\,|\,A_j=0$, so $\pi=\rho_{\pi|A_i}$. Therefore (1) is satisfied.

Lemma 3. *Let A be a* C*-*algebra which has a composition series* $\{I_j\}_{j=0,1,2,\cdots,n(<\infty)}$ (an increasing family of closed two-sided ideals I_j of A such that $I_0=(0)$ and $I_n=A$) *satisfying the following condition: if $0\leqq j\leqq n-1$, I_{j+1}/I_j is *-isomorphic to a product* C*-*algebra of a finite number, say n_j, of elementary* C*-*algebras. Then $\hat{A}$ consists of $\sum_{j=0}^{n-1}n_j$ elements.*

Proof. By Lemma 2 and 2.11.2 in [1], $\hat{A}^{I_1}$ consists of n_0 elements and $\hat{A}-\hat{A}^{I_1}_{I_0}=\hat{A}_{I_1}$, where $\hat{A}^I$ denotes the set of elements π of $\hat{A}$ such that $\pi(I)\neq(0)$. We assume $0<k<n$ and suppose that

$$\hat{A}-\bigcup_{0\leqq j\leqq k-1}\hat{A}^{I_{j+1}}_{I_j}=\hat{A}_{I_k}.$$

Then

$$\hat{A}-\bigcup_{0\leqq j\leqq k}\hat{A}^{I_{j+1}}_{I_j}=\hat{A}_{I_k}-\hat{A}^{I_{k+1}}_{I_k}=(\hat{A}-\hat{A}^{I_{k+1}})\cap\hat{A}_{I_k}$$
$$=\hat{A}_{I_{k+1}}\cap\hat{A}_{I_k}=\hat{A}_{I_{k+1}}.$$

Making use of the mathematical induction, we have

$$\hat{A}-\bigcup_{0\leqq j\leqq n-1}\hat{A}^{I_{j+1}}_{I_j}=\hat{A}_{I_n},\quad\text{so}\quad\hat{A}=\bigcup_{0\leqq j\leqq n-1}\hat{A}^{I_{j+1}}_{I_j}.$$

Now $\hat{A}^{I_{j+1}}_{I_j}$ consists of n_j elements, for $\hat{A}^{I_{j+1}}_{I_j}$ can be identified with $(I_{j+1}/I_j)^\wedge$ (cf. 2.11.2 in [1]). This completes the proof.

Lemma 3 raises the following

Proposition. *If a* C*-*algebra A has a composition series* $\{I_\rho\}_{0\leqq\rho\leqq\alpha}$ (an increasing family of closed two-sided ideals I_ρ of A indexed by the set of ordinals less than or equal to an ordinal α, such that $I_0=(0)$, $I_\alpha=A$ and if $\rho\leqq\alpha$ is a limit ordinal then $\bigcup_{\rho'<\rho}I_{\rho'}$ is dense in I_ρ) *satisfying the following condition:*

$$I_{\rho+1}/I_\rho \textit{ is elementary for all } \rho.$$

Then Card $\hat{A}=$Card α.

Proof. Since $I_0=(0)$, I_1 is elementary. Therefore $\hat{A}^{I_1}_{I_0}$ consists of an element π_0 and $\hat{A}-(\pi_0)=\hat{A}_{I_1}$. Suppose that $\hat{A}^{I_{\xi+1}}_{I_\xi}$ consists of an element π_ξ and

$$\hat{A}-\bigcup_{0\leqq\eta\leqq\xi}\pi_\eta=\hat{A}_{I_{\xi+1}}$$

for every ξ such that $0\leqq\xi<\nu\leqq\alpha$. Then, in case ν is an isolated number, we have

$$\hat{A}-\bigcup_{0\leqq\eta\leqq\nu-1}\pi_\eta=\hat{A}_{I_\nu}.$$

Since $I_{\nu+1}/I_\nu$ is elementary, $\hat{A}^{I_{\nu+1}}_{I_\nu}$ consists of an element π_ν and we have

$$\hat{A}-\bigcup_{0\leqq\eta\leqq\nu}\pi_\eta=\hat{A}_{I_{\nu+1}}.$$

In case ν is a limit number, suppose that there is an element π in $\hat{A}-\bigcup_{0\leqq\eta<\nu}\pi_\eta$. Suppose that there is an ordinal number $\xi<\nu$ such that $I_\xi\not\subset\operatorname{Ker}\pi$. For each element ρ of $\hat{A}-\bigcup_{0\leqq\eta\leqq\xi}\pi_\eta$, we have $\rho|I_{\xi+1}=0$. Since $\hat{A}-\bigcup_{0\leqq\eta<\nu}\pi_\eta\subset\hat{A}-\bigcup_{0\leqq\eta\leqq\xi}\pi_\eta$, π belongs to $\hat{A}-\bigcup_{0\leqq\eta\leqq\xi}\pi_\eta$, so $\pi|I_{\xi+1}=0$, hence $I_\xi\not\subset\operatorname{Ker}\pi$ is a contradiction. Therefore

$$\bigcup_{0\leqq\eta<\nu}I_\eta\subset\operatorname{Ker}\pi,\quad\text{so}\quad I_\nu\subset\operatorname{Ker}\pi.$$

Thus, when $\nu\neq\alpha$, we have

$$\hat{A}_{I_\nu}^{I_{\nu+1}}=(\pi_\nu)\quad\text{and}\quad\hat{A}-\bigcup_{0\leqq\eta\leqq\nu}\pi_\eta=\hat{A}_{I_{\nu+1}},$$

and when $\nu=\alpha$

$$\hat{A}-\bigcup_{0\leqq\eta<\nu}\pi_\eta=\phi.$$

Consequently, if $0\leqq\xi<\alpha$, then $\hat{A}_{I_\xi}^{I_{\xi+1}}=(\pi_\xi)$, $\hat{A}-\bigcup_{0\leqq\eta\leqq\xi}\pi_\eta=\hat{A}_{I_{\xi+1}}$ and $\hat{A}=\bigcup_{0\leqq\eta<\alpha}\pi_\eta$. Thus we have Card $\hat{A}=$ Card α.

Theorem. *A necessary and sufficient condition for a* C*-*algebra A to be a CCR-algebra whose dual $\hat{A}$ consists of a finite number, say n, of elements is that A is *-isomorphic to a product* C*-*algebra of n elementary* C*-*algebras.*

Proof. Suppose that A is a *CCR*-algebra and $\hat{A}$ consists of n elements $\pi_1,\pi_2,\cdots,\pi_n$, and let P_i be the kernel of π_i. These primitive ideals $P_1,P_2,\cdots,P_n$ are distinct maximal closed two-sided ideals of A (cf. 4.1.11 in [1]).
Let $P_{i_1},P_{i_2},\cdots,P_{i_{n-1}}$ be $n-1$ distinct elements of Prim(A), the ideal structure space with the Jacobson topology. Since A is a *CCR*-algebra, Prim(A) is a T_1-space (cf. 3.1.4 in [1]). Therefore $\{P_{i_1},P_{i_2},\cdots,P_{i_{n-1}}\}$ is a closed set of Prim(A), so

$$\{P\in\operatorname{Prim}(A)\,|\,P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}\subset P\}=\{P_{i_1},P_{i_2},\cdots,P_{i_{n-1}}\}.$$

Hence

$$P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}\neq(0)$$

and

$$P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}\not\subset P_m(m\neq i_1,i_2,\cdots,i_{n-1}).$$

Therefore $P_1\cap P_2\not\subset P_3$, and by the maximality of P_3 we have $P_1\cap P_2+P_3=A$. Thus any element x of P_1 may be written in the form of a sum $y+z$ where $y\in P_1\cap P_2$ and $z\in P_3$. Then $z=x-y\in P_1$, hence $P_1\subset P_1\cap P_2+P_1\cap P_3$. On the other hand $P_1\cap P_2+P_1\cap P_3\subset P_1\cap(P_2+P_3)\subset P_1$. Accordingly, we have $P_1=P_1\cap P_2+P_1\cap P_3$. Consequently, $A=P_2\cap P_3+P_1=P_2\cap P_3+P_1\cap P_2+P_1\cap P_3$. Suppose that for some $k(1<k<n-1)$ we have

$$A=\sum_{i_1,i_2,\cdots,i_k}P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k},$$

where the sum runs through all sets $\{i_1,i_2,\cdots,i_k\}$ which can be formed from k elements of $\{1,2,\cdots,k+1\}$. Now, for any such set $\{i_1,i_2,\cdots,i_k\}$,

$P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k}$ is a CCR-algebra, and its ideal structure space Prim $(P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k})$ consists of the following elements:

$$(P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k})\cap P_m(1\leqq m\leqq n,\ m\neq i_1,\ i_2,\ \cdots,\ i_k).$$

Therefore, these elements are all maximal closed two-sided ideals of $P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k}$ (cf. 4.1.11 in [1]), and by the same argument which we used just now, we have

$$P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k}=P_1\cap P_2\cap\cdots\cap P_{k+1}+P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_k}\cap P_{k+2}.$$

Therefore

$$A=\sum_{i_1,i_2,\cdots,i_{k+1}}P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{k+1}},$$

where the sum runs through all sets $\{i_1, i_2, \cdots, i_{k+1}\}$ which can be formed from $k+1$ elements of $\{1, 2, \cdots, k+2\}$. Making use of the mathematical induction, we have

$$A=\sum_{i_1,i_2,\cdots,i_{n-1}}P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}$$

where the sum runs through all sets $\{i_1, i_2, \cdots, i_{n-1}\}$ which can be formed from $n-1$ elements of $\{1, 2, \cdots, n\}$. On the other hand $P_1\cap P_2\cap\cdots\cap P_n=(0)$ (cf. 2.7.3 in [1]). Hence A is $*$-isomorphic to the product C*-algebra of n C*-algebras $(P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}})$. Consider an element $x\in P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}$ such that $\pi_m(x)=0$ where $m\neq i_1, i_2, \cdots, i_{n-1}$, Then $x\in P_1\cap P_2\cap\cdots\cap P_n$, so $x=0$. Hence the mapping $x\to\pi_m(x)$ from $P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}$ onto $LC(H_{\pi_m})$ is a $*$-isomorphism, namely $P_{i_1}\cap P_{i_2}\cap\cdots\cap P_{i_{n-1}}$ is an elementary C*-algebra. Consequently, A is $*$-isomorphic to a product C*-algebra of n elementary C*-algebras.

The sufficiency of the condition is easily seen by Lemma 2.

Corollary. *A* C*-*algebra A is a GCR-algebra whose dual $\hat{A}$ consists of a finite number, say n, of elements, if and only if A has a composition series $\{I_j\}_{j=0,1,\cdots,m(<\infty)}$ such that each I_{j+1}/I_j is $*$-isomorphic to a product* C*-*algebra of n_j elementary* C*-*algebras where $\sum_{j=0}^{m-1}n_j=n$.*

Proof. This corollary follows from Lemma 3 and the preceding theorem plus the following facts:

1) A C*-algebra A is a GCR-algebra if and only if A has a composition series $\{I_\rho\}_{0\leqq\rho\leqq\alpha}$ such that each $I_{\rho+1}/I_\rho$ is a CCR-algebra (cf. 4.3.4 in [1]).

2) $(I_{\rho+1}/I_\rho)^\wedge$ can be identified with $\hat{A}^{I_{\rho+1}}_{I_\rho}$ (cf. Proof of Lemma 3 for the notation of $A^{I_2}_{I_1}$).

3) $\hat{A}^{I_{\rho+1}}_{I_\rho}\cap\hat{A}^{I_{\rho'+1}}_{I_{\rho'}}=\varnothing$ for any pair ρ, $\rho'(\rho\neq\rho')$ and $\hat{A}=\bigcup_{0\leqq\rho<\alpha}\hat{A}^{I_{\rho+1}}_{I_\rho}$.

Remark 1. In the above corollary, the case $m=1$ and $n_0=n$ appears if and only if A is a CCR-algebra whose dual consists of n elements (cf. the above theorem).

2. We can restate the case $m=1$ and $n_0=n=1$ of the above corollary as follows:

A C*-algebra is elementary if and only if it is a *GCR*-algebra whose dual consists of a single element.

On the other hand, simple C*-algebra of type I is elementary [3] and vice versa (4.1.7 in [1]). Consequently the following three statements on a C*-algebra A are equivalent:

(i) A is elementary,

(ii) A is of type I and $\hat{A}$ consists of a single element,

(iii) A is of type I and simple.

3. If A is a separable C*-algebra with Card $\hat{A} \leqq \aleph_0$, $\hat{A}$ is a *GCR*-algebra by Lemma 1. Therefore the dual $\hat{A}$ of a separable C*-algebra A consists of a finite number n of elements, if and only if A has the same composition series that we gave in the preceding corollary. This result is an extension of the proposition in [5] which was refered to in the beginning of this paper.

References

[1] J. Dixmier: Les C*-algèbres et leurs représentations. Gauthier-Villars, Paris (1964).

[2] ——: Utilisation des facteurs hyperfinis dans la théorie des C*-algèbres. C. R. Acad. Sci. Paris, **258**, 4184–4187 (1964).

[3] ——: Simple C*-algebras. Lecture Note.

[4] J. Glimm: Type I C*-algebras. Ann. Math., **73**, 572–612 (1961).

[5] A. Rosenberg: The number of irreducible representations of simple rings with no minimal ideals. Amer. J. Math., **75**, 523–530 (1953).

[6] S. Sakai: On type I C*-algebras. Proc. A. M. S., **18**, 861–863 (1967).

[7] J. Tomiyama: A remark on representations of *CCR*-algebras (to appear).