133. *On Conservativity of Algebraic Function Fields*

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1. Let K be a field of algebraic functions of one variable over a field k of characteristic $p\neq 0$. Throughout this note, we assume that K is separable over k and k is algebraically closed in K. If the genus of K/k is invariant under any constant field extension of K/k, we say that K/k is *conservative*. E. Artin has proved that K/k is conservative if and only if for all finite purely inseparable constant field extensions $\tilde{K}/\tilde{k}$ of K/k, the genus of K/k is equal to the genus of $\tilde{K}/\tilde{k}$ (Chapter 15 of [1]).

Let K/k be as above, $M=\bigcup_{i=1}^{n} M_i$ a complete normal model of K/k, where $M_1, \cdots, M_n$ are affine models defined by affine k-algebras $A_1, \cdots, A_n$ respectively. Furthermore, we assume that each A_i is isomorphic to $k[X_{i1}, \cdots, X_{il_i}]/\mathfrak{a}_i$, where $k[X_{i1}, \cdots, X_{il_i}]$ is a polynomial ring and $\mathfrak{a}_i$ is a prime ideal of $k[X_{i1}, \cdots, X_{il_i}]$. In this note, we fix a normal complete model M and a set of equations for M, i.e., the union $\bigcup_{i=1}^{n} B_i$ where $B_i=\{F_{i1}(X), \cdots, F_{is_i}(X)\}$ is a basis of $\mathfrak{a}_i$. Let Ω be the set of all coefficients in the equations belonging to the set of equations for M, $\Delta=\{a_1, a_2, \cdots, a_m\}$ a p-basis of $k^p(\Omega)$ over k^p and let $\Delta^{p^{-1}}=\{a_1^{p^{-1}}, a_2^{p^{-1}}, \cdots, a_m^{p^{-1}}\}$. Then we have the following:

Theorem. *K/k is conservative if and only if the genus of K/k is equal to the genus of $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$.*

Remark. (1) We say that an algebraic function field $\tilde{K}/\tilde{k}$ is a constant field extension of K/k if $\tilde{K}=\tilde{k}K$ and K is free from $\tilde{k}$ over k. If we choose the above $a_i^{p^{-1}}(i=1, 2, \cdots, m)$ from a fixed complete field k^* which contains k, then we can construct the constant field extension $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ of K/k by the method of Chevalley [2].

(2) Let M and A_i $(i=1, 2, \cdots, m)$ be as stated above. Then the model of $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ defined by $k(\Delta^{p^{-1}})[A_i]$ $(i=1, 2, \cdots, n)$ is denoted by $M\otimes k(\Delta^{p^{-1}})$ (to prove Theorem, we shall consider this model $M\otimes k(\Delta^{p^{-1}})$ as a model over k). The geometric genus of M (resp. $M\otimes k(\Delta^{p^{-1}})$) is equal to the genus of K/k (resp. $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$) (cf. §6 of [4]).

(3) By a differential constant field for M (or K/k), we mean a field k_0 which satisfies the following three conditions:

(i) $k \supseteq k_0 \supseteq k^p$, $[k:k_0]<\infty$,

(ii) K^p and k_0 are linearly disjoint over k^p,

(iii) for any valuation ring $R_{\mathfrak{p}}$ of K, the differential module $M(R_{\mathfrak{p}}/k_0)$ of $R_{\mathfrak{p}}$ over k_0 is a free $R_{\mathfrak{p}}$-module (cf. [5]). If k itself is a differential constant field for K/k, then K/k is conservative by Theorem 4 of [5] (cf. Chapter III of [3] and Lemma 3 of [7]).

By making use of Remark 3 and Lemma (see § 2), we shall prove the above Theorem and some corollaries in § 3.

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2. Lemma. *Let K/k, $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$, M and $M\otimes k(\Delta^{p^{-1}})$ be as stated above. If the genus of K/k and the genus of $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ are equal to each other, then $M\otimes k(\Delta^{p^{-1}})$ is also normal.*

Proof. Let $R_{\mathfrak{p}}$ be the valuation ring belonging to a prime divisor $\mathfrak{p}$ of K/k and $S_{\mathfrak{P}}$ be the valuation ring of $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ which lies above $R_{\mathfrak{p}}$. If the genus of K/k and the genus of $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ are equal to each other, then by Theorem 20 of Chapter 15 of [1], we have $S_{\mathfrak{P}}=R_{\mathfrak{p}}\cdot k(\Delta^{p^{-1}})$. Since $k(\Delta^{p^{-1}})$ is a purely inseparable extension of k, $S_{\mathfrak{P}}=R_{\mathfrak{p}}\cdot k(\Delta^{p^{-1}})$ is the unique valuation ring of $K(\Delta^{p^{-1}})$ lying above $R_{\mathfrak{p}}$. Let A be one of the defining affine k-algebras of M and let $V(K)$ be the set of all valuation rings of K which contain A. Then the set $V(K(\Delta^{p^{-1}}))$ of all valuation rings of $K(\Delta^{p^{-1}})$ which contain $k(\Delta^{p^{-1}})[A]$ is equal to the set of all valuation rings of $K(\Delta^{p^{-1}})$ which lie above an element in $V(K)$. Since M is normal, the intersection of all elements in $V(K)$ is A. To prove Lemma, it is sufficient that the intersection of all elements in $V(K(\Delta^{p^{-1}}))$ is $k(\Delta^{p^{-1}})[A]$. Let c be an element of the intersection of all elements in $V(K(\Delta^{p^{-1}}))$. Then c is written in the form $c=\sum_{i=1}^{s} b_i w_i$ where $\{w_1, w_2, \cdots, w_s\}$ is a linear basis of $k(\Delta^{p^{-1}})$ over k and b_i $(i=1,2,\cdots,s)$ are elements in K. Since any element in $V(K(\Delta^{p^{-1}}))$ is of the form $\bigoplus_{i=1}^{s} R_{\mathfrak{p}}[w_i]$ (direct) where $R_{\mathfrak{p}}\in V(K)$ and since K is linearly disjoint from $k(\Delta^{p^{-1}})$ over k, b_i $(i=1,2,\cdots,s)$ are contained in A. Therefore, we have $c\in k(\Delta^{p^{-1}})[A]$. It is obvious that the intersection of all elements in $V(K(\Delta^{p^{-1}}))$ contains $k(\Delta^{p^{-1}})[A]$. That is, the intersection of all elements in $V(K(\Delta^{p^{-1}}))$ is $k(\Delta^{p^{-1}})[A]$.

3. Proof of Theorem. Assume that K/k and $K(\Delta^{p^{-1}})/k(\Delta^{p^{-1}})$ have the same genus. Then, the set of equations for M is also a set of equations for $M\otimes k(\Delta^{p^{-1}})$ over k by the proof of Lemma. From this fact, $k_0=k(\alpha_i, i\in I)$ where $(\alpha_i, i\in I)$ is a p-basis of k over $k^p(\Omega)$ is a common differential constant field for M and $M\otimes k(\Delta^{p^{-1}})$, the latter being considered as a model of $K(\Delta^{p^{-1}})/k$ (cf. Lemma 5 of [4] and Remark 3 of [6]). Furthermore, $\Delta=\{a_1, a_2, \cdots, a_m\}$ is a p-basis of k

over k_0. Since $M(S_{\mathfrak{P}}/k_0)$ is a free $S_{\mathfrak{P}}$-module,

$$k_0[S_{\mathfrak{P}}^p]=S_{\mathfrak{P}}\cap k_0\cdot K^p(\Delta)=R_{\mathfrak{p}}\cap k\cdot K^p \text{ by Theorem 1 of [5].}$$

On the other hand, we have

$$k_0[S_{\mathfrak{P}}^p]=k_0[k^p(\Delta)[R_{\mathfrak{p}}^p]]=k[R_{\mathfrak{p}}^p].$$

Then $k[R_{\mathfrak{p}}^p]=R_{\mathfrak{p}}\cap k\cdot K^p$ and $M(R_{\mathfrak{p}}/k)$ is a free $R_{\mathfrak{p}}$-module by Theorem 1 of [5]. Since K is separably generated over k, K^p and k are linearly disjoint over k^p. Therefore, k itself is a differential constant field for M. Hence K/k is conservative. The converse is obvious.

Corollary 1. *If there exists a normal complete model of K/k which is defined over k^p, then K/k is conservative.*

Corollary 2. *K/k is conservative if and only if K/k and kK^p/k have the same genus.*

Proof. If K/k is conservative, K^p/k^p is conservative. Because, K/k and K^p/k^p are isomorphic and have the same genus. Since kK^p/k is a constant field extension of K^p/k^p, the genus of K/k is equal to the genus of kK^p/k. Conversely, if the genus of K/k is equal to the genus of kK^p/k, K^p/k^p and kK^p/k have the same genus. Hence K^p/k^p is conservative by Theorem, i.e. K/k is conservative.

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