123. Korteweg-deVries Equation. II

Finite Difference Approximation

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In the preceding note [1] we announce the fact that the Cauchy problem for the KdV equation has global smooth solution uniquely for any sufficiently smooth initial data. Here we show the existence of the computable approximate solutions of the Cauchy problem for the KdV equation by the method of finite difference schemes.

To approximate the Cauchy problem for the KdV equation

$$(1)\qquad D_t u = uDu + D^3 u \qquad (x,t)\in R^1\times(0,\infty) \qquad u=u(x,t)$$

$$(2)\qquad u(x,0)=f(x) \qquad x\in R^1 \qquad D_t=\frac{\partial}{\partial t},\quad D=\frac{\partial}{\partial x}$$

We propose following special implicit finite difference scheme

$$(3)\qquad D^+u_j^n = Eu_j^{n+1}D_0u_j^{n+1} + D_+D_-^2u_j^{n+1} \qquad j=0,\pm1,\pm2,\cdots \quad n=0,1,2,\cdots$$

$$(4)\qquad u_j^0=f_j \qquad j=0,\pm1,\pm2,\cdots$$

Here we use the following notations

$$u_j^n=u(jh, n\Delta t) \quad h:\text{ mesh-width} \quad \Delta t=\lambda h^3:\text{ time-step}$$
$$\lambda=\text{const.}:\text{ mesh ratio}$$

$$D^+u_j^n=\frac{1}{\Delta t}(u_j^{n+1}-u_j^n)$$

$$D_+u_j^n=\frac{1}{h}(u_{j+1}^n-u_j^n)$$

$$D_-u_j^n=\frac{1}{h}(u_j^n-u_{j-1}^n)$$

$$D_0u_j^n=\frac{1}{2h}(u_{j+1}^n-u_{j-1}^n)$$

$$Eu_j^n=\frac{1}{3}(u_{j+1}^n+u_j^n+u_{j-1}^n)$$

To solve (1) with respect to u_j^{n+1} we use following iteration

$$(5)\qquad u_j^{n+1-\frac{1}{m+1}}=u_j^n+\lambda h^2Eu_j^{n+1-\frac{1}{m}}Fu_j^{n+1-\frac{1}{m}}+8\lambda Gu_j^{n+1-\frac{1}{m}} \qquad m=1,2,3,\cdots$$

Here

$$Fu_j^n=\frac{1}{2}(u_{j+1}^n-u_{j-1}^n)$$

$$Gu_j^n=\frac{1}{8}(u_{j+1}^n-3u_j^n+3u_{j-1}^n-u_{j-2}^n)$$

There exist following a priori estimate for the solutions $u_h=u_j^n$ of (3) and (4).

Proposition 1. $\|u^n\|_h \leq \|u^{n-1}\|_h \leq \cdots \leq \|f\|_h$ for $^\forall n \geq 1$

Here

$$\|u^n\|_h^2 = \sum_j h\,|u_j^n|^2$$

Proposition 2. *For any non-negative integer k, if we take*

$$t_1 = 1/\sup_{0\leq h\leq h_0} \alpha(\|f\|_h, \cdots, \|D_+^3 f\|_h)$$

$$t_k = 1/\sup_{0\leq h\leq h_0} (1+k)\beta(\|f\|_h, \cdots, \|D_+^3 f\|_h)$$

Then there exist for $i+j\leq k$, a priori estimate

$$\|D^{+i}D_+^{3j}u^n\|_h \leq P_k(\|f\|_h, \cdots, \|D_+^{3k}f\|_h)$$

for $(n+k)\Delta t\leq t_1,\ t_k$. Here α, β and P_k are polynomials with positive coefficients.

Using Propositions 1 and 2 (in the case of $k=1$) we can show

Theorem 1. *If*

$$\lambda < \frac{1}{8}, \qquad h \leq \frac{1-8\lambda}{\sqrt{\lambda}} 1 \Big/ \sup_{0\leq h\leq h_0} \gamma(\|f\|_h, \cdots, \|D_+^3 f\|_h).$$

Here γ is a polynomial with positive coefficients. Then the iteration scheme (5) converges for $n\Delta t\leq t_1$. Therefore the implicite scheme (3) and (4) is uniquely solvable for any sufficiently smooth initial data $f(x)$.

Next we show this computable solution $u_h=u_j^n$ really approximates the true solution of the Cauchy problem for the KdV equation corresponding the same initial data $f(x)$. First we consider the smooth interpolation of u_h using the discrete Fourier transformation with respect to t and x. We modify u_j^n by multiplying $\rho(t)\in\mathcal{D}$ which is equal to 1 for $0\leq t\leq \frac{1}{2}\min(t_1, t_k)$ and equal to 0 for $t\geq \min(t_1, t_k)$. Next we extend u_j^n for $n<0$ in such a way as the estimate

$$\sum_{i+j\leq k} \|D^{+i}D_+^{3j}u^n\|_h \leq \text{const.} \sup_{m\geq 0} \sum_{i+j\leq k} \|D^{+i}D_+^{3j}u^m\|_h$$

holds for $^\forall n<0$. This modified solution we also write $u_h=u_j^n$. The estimate in Proposition 2 remains true for any n for this modified u_h.

Definition.

$$(6) \qquad \tilde{u}_h(x, t) = \int_{\substack{|2\xi h|\leq 1\\ |2\tau\Delta t|\leq 1}} e^{2\pi i(\xi x+\tau t)}\,\hat{u}_h(\xi, \tau)\,d\xi d\tau$$

Here

$$\hat{u}_h(\xi, \tau) = \sum_{j,n} e^{-2\pi i(\xi jh+\tau n\Delta t)} u_j^n h\Delta t$$

It is obvious that

$$\tilde{u}_h(jh, n\Delta t) = u_j^n$$

Therefore we call $\tilde{u}_h(x, t)$ the smooth interpolation of $u_h=u_j^n$.

Lemma 1.

$$\|D^{+i}D_+^j u_h\|_{L_h^2(R^2)} \leq \|D_t^i D^j \tilde{u}_h\|_{L^2(R^2)} \leq \left(\frac{\pi}{2}\right)^{i+j} \|D^{+i}D_+^j u_h\|_{L_h^2(R^2)}$$

for any i,j. Here

$$\|u_h\|^2_{L^2_h(R^2)} = \sum_{j,n} h\Delta t\,|u^n_j|^2$$

By Lemma 1 and Proposition 2 we have following estimate for $\tilde{u}_h$.

Proposition 3. *For any non-negative integer k, there exist following estimate*

$$\|D^i_t D^{3j}\tilde{u}_h\|_{L^2(R^2)} \leq \text{const. } P_k(\|f\|_h, \cdots, \|D^{3k}_+ f\|_h)$$

for $i+j\leq k$.

From Proposition 3 using Sobolev's lemma we have

$$\begin{aligned}|D^i_t D^{3j}\tilde{u}_h|_{\mathcal{B}^1} &\leq \text{const. } \|D^i_t D^{3j}\tilde{u}_h\|_{\mathcal{E}^3_{L^2(R^2)}}\\ &\leq \text{const. } P_{k+3}(\|f\|_h, \cdots, \|D^{3(k+3)}_+ f\|_h)\end{aligned}$$

for $i+j\leq k$.

By Ascoli-Arzela's theorem we conclude

Theorem 2. *For any non-negative integer k, if we take*

$$T_k = \frac{1}{2}\min(t_1, t_{k+3})$$

Then for $i+j\leq k$ there exists a subsequence $D^i_t D^{3j}\tilde{u}_{h'}$ of $D^i_t D^{3j}\tilde{u}_h$ such that $D^i_t D^{3j}\tilde{u}_{h'}$ converges to $D^i_t D^{3j}u$ as $h'\to 0$ compact uniformly on $R^1\times[0, T_k]$. Where u is the true solution of the Cauchy problem for the KdV equation.

Concerning error estimate we have the following result. In this case we regard the approximate solution $u_h = u^n_j$ as a step function which takes the constant value u^n_j on each interval $\left(\left(j-\frac{1}{2}\right)h, \left(j+\frac{1}{2}\right)h\right)$ $j=0, \pm 1, \pm 2, \cdots$ respectively.

Theorem 3. *Let $u(x, t)$ be the true solution of the Cauchy problem for the* KdV *equations* (1) *and* (2). *Let u^n_j be the solution of the implicit scheme* (3) *and* (4) *corresponding the same initial data $f(x)$.*

Then we have the following error estimate

$$\begin{aligned}\|u(x, t) - u^n_j\|_{L^2(R^1)} &\leq 0(|t - n\Delta t|)Q(\|f\|, \cdots, \|D^3 f\|)\\ &\quad + 0(h)R(\|f\|, \cdots, \|D^7 f\|)\end{aligned}$$

for $0\leq t,\ n\Delta t\leq t_1$. Here Q and R are polynomials with positive coefficients.

References

[1] Kametaka, Y.: Korteweg-deVries equation. I. Global existence of smooth solutions. Proc. Japan. Acad., **45**, 552-555 (1969).

[2] Sjöberg, A.: On the Korteweg-deVries equation. Uppsala Univ., Dept. of Computer Sci., Report (1967).

[3] Zabusky, N., and Kruskal, M.: Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Letters, **15**, 240–243 (1969).