116. *Stability Problems on Difference and Functional-differential Equations*

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In this paper, we shall show some theorems on the stability problems of difference and functional-differential equations by means of two methods, one of which is directly dependent on the forms of equations and the other is to make use of Lyapunov functionals.

1. Definition of stability. Before stating the definitions of stability, it is convenient to introduce two norms and a family of functions. Let $f(t)$ be a function with ith component $f_i(t)$ $(i=1, \cdots, n)$ defined for an interval I. Then we define two norms such that $|f(t)|=\max_{1\leqq i\leqq n}|f_i(t)|$ for any $t\in I$ and $\|f\|=\sup_{t\in I}|f(t)|$. Let $\mathcal{S}$ be a family of functions which have the following properties:

(i) every function in $\mathcal{S}$ is defined for $s\in[-1, 0)$;

(ii) every function in $\mathcal{S}$ has a limit as $s\to-0$.

Now we shall consider a difference equation[1)]

$$x(t)=f(t, x(t-1)), \tag{1}$$

where $f(t, x)$ is defined for $t_0+k\leqq t<t_0+k+1$ $(k=0, 1, \cdots)$ and $|x|<H$, for any fixed x the limit as $t\to t_0+k-0$ $(k=1, 2, \cdots)$ exists, and $f(t, 0)=0$ for any fixed t. Then we suppose that the difference equation (1) has a solution $x(t)$ for $t\geqq t_0$ such that $|x(t)|<H$ under the initial condition

$$x(t)=\varphi(t), \quad t_0-1\leqq t<t_0. \tag{2}$$

Here the initial function $\varphi(t)$ is a given function defined for $t\in[t_0-1, t_0)$, has a limit as $t\to t_0-0$, and satisfies $\|\varphi(t_0+s)\|<H$, where the norm is defined as before, if we consider the function $\varphi(t_0+s)$ to be in $\mathcal{S}$ as s varies over the interval $[-1, 0)$. If we denote by $x(t, t_0, \varphi)$ the solution of (1) with the initial condition (2), the stability of the trivial solution of (1) will be defined following those of functional-differential equations.

Definition 1. The trivial solution of (1) is said to be *stable* if for any given $\varepsilon>0$ there exists a $\delta(\varepsilon, t_0)$ such that $\|\varphi(t_0+s)\|<\delta(\varepsilon, t_0)$ implies $|x(t, t_0, \varphi)|<\varepsilon$ for any $t\geqq t_0$.

1) In this paper, every equation will be treated in the n-dimensional vector space.

Definition 2. The trivial solution of (1) is said to be *uniformly stable* if for any given $\varepsilon>0$ there exists a $\delta(\varepsilon)$ such that $\|\varphi(t_0+s)\|<\delta(\varepsilon)$ implies $|x(t, t_0, \varphi)|<\varepsilon$ for any $t\geqq t_0$.

Definition 3. The trivial solution of (1) is said to be *asymptotically stable* if it is stable and there exists a $\delta_0(t_0)$ such that $\|\varphi(t_0+s)\|<\delta_0(t_0)$ implies $\lim_{t\to\infty}|x(t, t_0, \varphi)|=0$.

Definition 4. The trivial solution of (1) is said to be *uniformly asymptotically stable* if it is uniformly stable and there exist a constant δ_0 and $T(\varepsilon)$ for any given $\varepsilon>0$ such that $\|\varphi(t_0+s)\|<\delta_0$ implies $|x(t, t_0, \varphi)|<\varepsilon$ for any $t\geqq t_0+T(\varepsilon)$.

2. Uniformly asymptotic stability of perturbed difference systems. Let $f(t, x)$ have the same properties as before and $A(t)$ be an $n\times n$ matrix having the same properties as $f(t, x)$ with respect to t and $\det A(t)\neq 0$ for any $t\geqq t_0$. Then it is supposed that the perturbed difference system

$$x(t)=A(t)x(t-1)+f(t, x(t-1)) \tag{3}$$

has a solution $x(t, t_0, \varphi)$ for $t\geqq t_0$ such that $|x(t, t_0, \varphi)|<H$ under the initial condition (2). Before stating the theorems, we have to prepare two lemmas.

Lemma 1.[2] *If the inequality*

$$u_m\leqq g_m+\sum_{k=0}^{m-1} K_k u_k \quad (m=0, 1, \cdots)$$

is satisfied, where $K_m\geqq 0$ $(m=0, 1, \cdots)$, we obtain an estimation for u_m such that

$$u_m\leqq g_m+\sum_{k=0}^{m-1} K_k g_k \prod_{\nu=k+1}^{m-1}(1+K_\nu) \quad (m=1, 2, \cdots).^{3)}$$

If $g_m\geqq 0$ $(m=0, 1, \cdots)$, we obtain

$$u_m\leqq g_m+\sum_{k=0}^{m-1} K_k g_k \exp\left(\sum_{\nu=k+1}^{m-1} K_\nu\right) \quad (m=1, 2, \cdots).$$

If $g_m\equiv c\geqq 0$ $(m=1, 2, \cdots)$ is constant and $K_m\geqq 0$, we obtain

$$u_m\leqq c\prod_{k=0}^{m-1}(1+K_k)\leqq c\exp\left(\sum_{k=0}^{m-1} K_k\right) \quad (m=1, 2, \cdots).$$

Lemma 2.[4] *Let $X(t, t_0)$ be a fundamental matrix of the homogeneous system $x(t)=A(t)x(t-1)$ such that $X(t_0, t_0)=E$ (the unit matrix). If the trivial solution of it is uniformly asymptotically stable, there exist positive constants B and α such that $\|X(t, t_0)\|\leqq B\exp(-\alpha(t-t_0))$ for any $t\geqq t_0$. Here $\|X\|$ represents the norm of X which is considered to be a linear operator.*

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2) This result corresponds to Gronwall's inequality in the theory of differential equations.

3) As usual, it is supposed that $\sum_{k=r}^{r-1} b_k=0$ and $\prod_{k=r}^{r-1} c_k=1$ for any b_k, c_k and integer r.

4) This lemma is proved as in the theory of functional-differential equations.

By means of two lemmas above, we can establish the following results.

Theorem 1. *In the equation* (3), *suppose that for sufficiently small constant c the function* $f(t, x)$ *satisfies an inequality* $|f(t, x)| \leqq c|x|$ *for* $t \geqq t_0$ *and* $|x| < h (\leqq H)$. *Then, if the trivial solution of the homogeneous system* $x(t) = A(t)x(t-1)$ *is uniformly asymptotically stable, the trivial solution of* (3) *is also uniformly asymptotically stable.*

Theorem 2. *In the equation* (3), *suppose that the function* $f(t, x)$ *satisfies an inequality* $|f(t, x)| \leqq \beta(t)|x|$ *for* $t \geqq t_0$ *and* $|x| < h (\leqq H)$, *where* $\beta(t)$ *is defined for* $t_0 + k \leqq t < t_0 + k + 1$ $(k = 0, 1, \cdots)$ *and the infinite series*

$$\sum_{k=1}^{\infty} \sup_{s \in [-1, 0)} \beta(t_0 + k + s)$$

is convergent. Then, if the trivial solution of the homogeneous system $x(t) = A(t)x(t-1)$ *is uniformly asymptotically stable, the trivial solution of* (3) *is also uniformly asymptotically stable.*

3. Applications of Lyapunov functionals to difference systems. It is well known that Lyapunov's V-functions play an important role in the stability theory of differential equations. In the following, four theorems using V-functionals will be stated for difference systems. Three functions $a(r)$, $b(r)$, and $c(r)$ will always be defined for $0 \leqq r < \infty$, continuous, strictly monotone increasing, and $a(0) = b(0) = c(0) = 0$.

Theorem 3. *Suppose that for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *there exists a functional* $V[t, \psi]$ *which has the following properties*:

(i) *for any given* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon, t_0)$ *such that* $\|\varphi(t_0 + s)\| < \delta(\varepsilon, t_0)$ *implies* $V[t_0, \varphi(t_0 + s)] < \varepsilon$;

(ii) *for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *the inequality* $a(\|\psi\|) \leqq V[t, \psi]$ *is satisfied*;

(iii) *for any solution* $x(t, t_0, \varphi)$ *of* (1) *the inequality* $\Delta V[t_0 + m, x(t_0 + m + s, t_0, \varphi)] \leqq 0$ *is satisfied for* $m = 0, 1, \cdots$.

Then the trivial solution of (1) *is stable.*

Theorem 4. *Suppose that for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *there exists a functional* $V[t, \psi]$ *which has the following properties*:

(i) *for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *the inequality* $a(\|\psi\|) \leqq V[t, \psi] \leqq b(\|\psi\|)$ *is satisfied*;

(ii) *for any solution* $x(t, t_0, \varphi)$ *of* (1) *the inequality* $\Delta V[t_0 + m, x(t_0 + m + s, t_0, \varphi)] \leqq 0$ *is satisfied for* $m = 0, 1, \cdots$.

Then the trivial solution of (1) *is uniformly stable.*

Theorem 5. *Suppose that for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *there exists a functional* $V[t, \psi]$ *which has the following properties*:

(i) *for any given* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon, t_0)$ *such that* $\|\varphi(t_0 + s)\| < \delta(\varepsilon, t_0)$ *implies* $V[t_0, \varphi(t_0 + s)] < \varepsilon$;

(ii) *for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *the inequality* $a(\|\psi\|) \leqq V[t, \psi]$ *is satisfied*;

(iii) *for any solution* $x(t, t_0, \varphi)$ *of* (1) *the inequality* $\Delta V[t_0 + m, x(t_0 + m$

$+ s, t_0, \varphi)] \leqq - c(V[t_0 + m + s, x(t_0 + m + s, t_0, \varphi)]$ *is satisfied for any* $m=0, 1, \cdots$.

Then the trivial solution of (1) *is asymptotically stable.*

Theorem 6. *Suppose that for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *there exists a functional* $V[t, \psi]$ *which has the following properties:*

(i) *for any* $t \geqq t_0$ *and* $\psi \in \mathcal{S}$ *the inequality* $a(\|\psi\|) \leqq V[t, \psi] \leqq b(\|\psi\|)$ *is satisfied;*

(ii) *for any solution* $x(t, t_0, \varphi)$ *of* (1) *the inequality* $\Delta V[t_0+m, x(t_0+m+s, t_0, \varphi)] \leqq -c(\|x(t_0+m+s, t_0, \varphi)\|]$ *is satisfied for* $m=0, 1, \cdots$.

Then the trivial solution of (1) *is uniformly asymptotically stable.*

Remarks 1. In case where t varies over a set of discrete points, $\{t_0+k\}_{k=0}^{\infty}$, the corresponding results to four theorems just stated above will be established by means of the same methods with the following replacements such that

(i) the norm $\|\cdots\|$ is replaced by the norm $|\cdots|$;

(ii) initial function is replaced by initial value;

(iii) a functional $V[t, \psi]$ is replaced by a function of t and x, $V(t, x)$;

(iv) in the hypothesis (i) of Theorems 3 and 5, the function $V(t, x)$ is continuous for $t \geqq t_0$ and $|x| < H$, and $V(t, 0)=0$ uniformly in t.

2. If we are concerned with the functional-differential equation

$$\dot{x}(t)=f(t, x(t+s)) \tag{4}$$

under the initial condition

$$x(t)=\begin{cases}\varphi(t), & t \in [t_0-1, t_0),\\ x^0, & t=t_0,\end{cases}$$

where $f(t, x(t+s))$ is a functional of $x(t+s)$ as s varies over an interval $[-1, 0)$, and if we define a function $\varphi^*(t)$ such that

$$\varphi^*(t)=\begin{cases}\varphi(t), & t \in [t_0-1, t_0),\\ x^0, & t=t_0,\end{cases}$$

the definitions of stability for the system (4) will be defined as before only except with the replacement of $\varphi(t)$ by $\varphi^*(t)$. Then we obtain the same Theorems 3-6 with the same replacement as above.