114. A Note on Variation Theory

Dedicated to Professor Atuo Komatu on his 60th birthday

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This note is a contribution to the study of variation problem and is intended to give a new version of the classical variation theory, due essentially to Lusternik-Schnirelmann.

We assume throughout this note that a space X is Hausdorff and locally connceted in all dimensions, that is, for every point $x \in X$ there is a neighbourhood U of x such that $H_*(U, Z) = H_*(x, Z)$, where $H_*(\ , Z)$ stands for integral homology group.

Let φ be a non-negative continuous function on X and let f be a map of X into itself. The triple $(X, \varphi ; f)$ is said to be a variation problem of type C (over a field k), if X, φ and f satisfy the following conditions:

A) $\varphi(f(x)) \leqq \varphi(x)$ for any $x \in X$.
B) $\varphi(f(x)) = \varphi(x)$ implies $\varphi(f \circ f(x)) = \varphi(x)$.
C) $f_* = \mathrm{id}: H_*(X, k) \to H_*(X, k)$.

Given a variation problem $(X, \varphi ; f)$, we define for a compact set $A \subset X$ as follows:

$$\varphi(A) = \sup_{a \in A} \varphi(a), \qquad |A| = \inf_n \varphi(f^n(A))$$
$$\gamma = \{x \in X \mid \varphi(f(x)) = \varphi(x)\}$$
$$F(A) = \overline{\bigcup_n f^n(A)}.$$

A point x in γ is called a φ-stationary point.
For $Y \subset X$, let

$$Y(a) = \{y \in Y \mid \varphi(y) = a\},$$
$$Y([a, b]) = \{y \in Y \mid a \leqq \varphi(y) \leqq b\},$$
$$Y((-\infty, b]) = \{y \in Y \mid \varphi(y) \leqq b\}.$$

Existence lemma. *Let $(X, \varphi ; f)$ be a variation problem of type C and let A be a compact subset of X such that $F(A) \cap X([|A|, \infty))$ is compact.*
Then

$$\gamma(|A|) \cap F(A) \neq \phi.$$

Proof. Take $a_n \in A$ so that

$$\varphi(f^n(a_n)) = \varphi(f^n(A)) \geqq |A|$$

and set $y_n = f^n(a_{n+1})$. Then the inequalities

$$\varphi(y_n) \geqq \varphi(f(y_n)) = \varphi(f^{n+1}(a_{n+1})) \geqq |A|$$

imply that $y_n \in F(A) \cap X([\,|A|, \infty))$. Now we choose a convergent subsequence $\{y_m\}$ of $\{y_n\}$. Setting $y=\lim y_m$, we clearly have $y \in F(A)$ and that

$$\begin{aligned}\varphi(f(y)) &= \varphi(\lim f(y_m)) = \lim \varphi(f(y_m)) \\ &= \lim \varphi(f^{m+1}(A)) = |A|, \\ \varphi(y) &= \lim \varphi(y_m) = \lim \varphi(f^m(a_{m+1})) \\ &\leqq \lim \varphi(f^m(A)) = |A|.\end{aligned}$$

Hence $\varphi(f(y)) = |A| \geqq \varphi(y)$.

Since $\varphi(f(y)) \leqq \varphi(y)$, we see that

$$\varphi(f(y)) = \varphi(y) = |A|$$

and

$$y \in \gamma(|A|) \cap F(A).$$

Deformation lemma. *Let $(X, \varphi; f)$ be a variation problem of type C and let A be a compact set in X.*
Then:

1) *Given $\varepsilon > 0$, there exists an integer $n_1 = n_1(\varepsilon, A)$ such that*

$$\varphi(f^n(A)) < |A| + \varepsilon \qquad \text{if } n \geqq n_1.$$

2) *If $F(A) \cap X([\,|A|, \infty))$ is compact, then for each open set $U \supset \gamma(|A|) \cap F(A)$ there exists an integer $n_2 = n_2(U, A)$ such that*

$$f^n(A) \subset U \cup X((-\infty, |A|)) \textit{ if } n \geqq n_2.$$

Proof. Part 1) follows immediately from the definition of $|A|$.

For the proof of 2), we need the following

Sublemma. *Let $(X, \varphi; f)$ be a variation problem of type C, then for any compact set A of X, we have*

$$\gamma \cap F(A) \subset (F(A) \cap \gamma)((-\infty, |A|]).$$

Proof of sublemma. Take $x \in F(A) \cap \gamma$ and let $\{f^{n(i)}(a_{n(i)})\}(a_n \in A)$ be a convergent sequence to x.

Case 1. If $\{n(i)\}$ is not bounded, then

$$\varphi(x) = \lim \varphi(f^{n(i)}(a_{n(i)})) \leqq \lim \varphi(f^{n(i)}(A)) = |A|.$$

Case 2. If $\{n(i)\}$ is bounded, then since $f^n(A)$ is compact, there is an integer n_0 such that $x \in f^{n_0}(A)$.

Set $x = f^{n_0}(a)(a \in A)$, then we have that

$$\varphi(x) = \varphi(f^k(x)) = \varphi(f^{n_0+k}(a)), \text{ for each } k > 0,$$

hence

$$\varphi(x) = \varphi(f^{n_0+k}(a)) \leqq \varphi(f^{n_0+k}(A)) \downarrow |A|.$$

The result follows by passing to the limit.

We now proceed to the proof of part 2) of the deformation lemma. Since $f(F(A)) \subset F(A)$ and $f(\gamma(|A|)) \subset \gamma(|A|)$, we have

$$f(\gamma(|A|) \cap F(A)) \subset \gamma(|A|) \cap F(A) \subset U.$$

Hence there is an open set V such that

$$\gamma(|A|) \cap F(A) \subset V \text{ and } f(V) \subset U.$$

By Sublemma we obtain

$$\gamma \cap F(A) \cap X([\,|A|, \infty)) = \gamma(|A|) \cap F(A),$$

from which it follows that a non negative function $\psi(x)=\varphi(x)-f(\varphi(x))$ $(x\in X)$ does not vanish on $F(A)\cap X([|A|,\infty))$ except on $\gamma(|A|)\cap F(A)$, thus ψ attains the minimum $\delta>0$ on the compact set $V^c\cap F(A)\cap X([|A|,\infty))$.

Now take $n_1=n_1(\delta)$ so that $n\geqq n_1$ implies that

$$\varphi(f^n(A))\leqq|A|+\delta/2.$$

then we easily see that

$$f\circ f^n(A)\subset U\cup X((-\infty,|A|)).$$

Given a singular chain σ (with coefficients in k) on X, let $|\sigma|$ denote $|Car\ \sigma|$. We then define the height $|\alpha|$ of a homology class $\alpha\in H_*(X;k)$ by

$$|\alpha|=\inf\{|\sigma|\,|\,[\sigma]=\alpha\}.$$

Furthermore, we denote by i, j the inclusion maps associated with pair (X, Y):

$$i: Y\to X,\qquad j: X\to(X, Y).$$

Lemma 1. *Let $(X, \varphi; f)$ be a variation problem of type C, then*:

1) i) *If $\alpha\in Ker\,(j_*: H_*(X)\to H_*(X, X((-\infty, a)))$, then $|\alpha|<a$,*
   ii) *If $\alpha\in Ker\,(j_*: H_*(X)\to H_*(X, X((-\infty, a]))$, then $|\alpha|\leqq a$.*
2) i) *If $|\alpha|<a$ $(a\geqq 0)$, then* $\alpha\in Ker\,(j_*: H_*(X)\to H_*(X, X((-\infty, a)))$,
   ii) *If $|\alpha|\leqq a$ $(a\geqq 0)$, then for any $\varepsilon>0$* $\alpha\in Ker(j_*: H_*(X)\to H_*(X, X(-\infty, a+\varepsilon))$

Definition. A homology class $\alpha\in H_*(X;k)$ is said to be *subordinate* to $\beta\in H_*(X;k)$ and is denoted by $\alpha\prec\beta$, if there exists a cohomology class $\xi\in H^m(X;k)$ such that

$$\alpha=\beta\cap\xi\qquad\text{with } m=\dim\xi>0.$$

Also a cycle σ is said to be subordinate to a cycle τ if $[\sigma]$ is subordinate to $[\tau]$.

Lemma 2. *Let $(X, \varphi; f)$ be a variation problem of type C and assume that $\alpha\prec\beta$. Then we have $|\alpha|\leqq|\beta|$.*

Proof. Let $\alpha=\beta\cap\xi$, $\xi\in H^*(X;k)$ and take a cycle τ of β and a cocycle Ξ of ξ. Setting $\sigma=\tau\cap\Xi$, we find that $\alpha=[\sigma]$ and $Car\ \sigma\subset Car\ \tau$. From this it follows that

$$|\alpha|\leqq|\sigma|=|Car\ \sigma|\leqq|Car\ \tau|.$$

thus the result follows.

Definition. A homology class $\alpha\in H_*(X, k)$ is said to be *realizable*, if there exists a cycle σ in X such that $\alpha=[\sigma]$ with $|\alpha|=|\sigma|$.

Proposition. *Let $(X, \varphi; f)$ be a variation problem of type C. If $\alpha\in H^*(X;k)$ is realizable by a cycle σ satisfying the following properties*:

1) *$F(Car\ \sigma)\cap X([|\alpha|,\infty))$ is compact,*

2) *There exists an open set U such that*

$$U \supset \gamma(|\alpha|) \cap F(Car\ \sigma) \text{ and } H^m(U\,;k)=0 \quad (\text{for any } m>0).$$

Then for any $\beta \prec \alpha$, it holds that $|\beta| < |\alpha|$.

Proof. Let $\beta = \alpha \cap \xi$, $\xi \in H^*(X)$. The inclusion maps associated with pairs $(X, U \cup J)$, (X, U), (X, J) are denoted by

$$\begin{array}{llll} i: & U \cup J \to X, & j: & X \to (X, U \cup J) \\ i_1: & U \to X, & j_1: & X \to (X, U) \\ i_2: & J \to X, & j_2: & X \to (X, J), \end{array}$$

where $J = X(-\infty, |\alpha|)$. This leads to the commutative diagram

$$\begin{array}{ccccc} H_*(U \cup J) & & H^*(U) & & H_*(J) \\ \downarrow i_* & & \uparrow i_1^* & & \downarrow i_{2*} \\ H_*(X) & \otimes & H^*(X) & \xrightarrow{\cap} & H_*(X) \\ \downarrow j_* & & \uparrow j_1^* & & \downarrow j_{2*} \\ H_*(X, U \cup J) & \otimes & H^*(X, U) & \xrightarrow{\cap} & H_*(X, J). \end{array}$$

Since $i_1^*(\xi)=0$, there is $\tilde{\xi} \in H^*(X, U)$ such that $j_1^*(\tilde{\xi})=\xi$. From the deformation lemma there is $\tilde{\alpha} \in H_*(U \cup J)$ such that $i_*(\tilde{\alpha})=\alpha$. By naturality of the cap product, we have

$$\begin{aligned} j_{2*}(\beta) &= j_{2*}(\alpha \cap \xi) = j_{2*}(\alpha \cap j_1^*(\tilde{\xi})) \\ &= j_{2*}(\alpha) \cap \tilde{\xi} = j_* i_*(\tilde{\alpha}) \cap \tilde{\xi} = 0. \end{aligned}$$

The result then follows from Lemma 1.

Theorem. *Let $(X, \varphi\,; f)$ be a variation problem of type C. For fixed a, suppose that $F(A) \cap X([a, \infty))$ is compact for any compact set A.*

If there is a sequence $\alpha_1, \alpha_2, \cdots \alpha_l$ of homology classes of X such that

1) $\alpha_i \prec \alpha_{i+1}$, $\quad i=1, 2, \cdots l-1$

2) $\alpha_i \notin Ker\ (j_*\,; H_*(X) \to H_*(X, X(-\infty, a)))$.

Then we have

$$\#\{\gamma([a, \infty))\} \geqq l.$$

Proof. Suppose some α_i be not realizable. Then we can choose a sequence $\{\sigma_i^n\}$ of cycles such that $\alpha_i = [\sigma_i^n]$ $(n=1, 2 \cdots)$ and

$$|Car\ \sigma_i^n| > |Car\ \sigma_i^{n+1}| \downarrow |\alpha_i|.$$

From the existence lemma it follows that

$$\gamma(|Car\ \sigma_i^n|) \neq \phi, \qquad \text{for all } n.$$

Hence we have $\#(\gamma([a, \infty))) \geqq \#(\sum \gamma(Car\ \sigma_i^n)) = \infty$.

Therefore we may restrict ourselves to the case where each α_i is realizable. Now suppose $|\alpha_i| = |\alpha_{i+1}|$ for some i. Then from the proposition we have that for any open set $U \supset \gamma(|\alpha_i|)$,

$$H^p(U) \neq 0,\ p = \dim \alpha_{i+1} - \dim \alpha_i.$$

Hence we see that $\dim U \geqq p$ and that

$$\#(\gamma([a, \infty))) \geqq \#(\gamma(|\alpha_i|)) = \infty.$$

Now it remains only the case where

$$a<|\alpha_1|<|\alpha_2|\cdots<|\alpha_l|.$$

Since by the existence lemma $\gamma(|\alpha_i|)\neq\phi$ for each i, it follows that

$$\sharp(\gamma([a,\infty)))\geqq\sharp(\sum_{i=1}^{l}\gamma(|\alpha_i|))\geqq l$$

This completes the proof.

Bibliography

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