152. On Tensor Products of Operators

By Takayuki FURUTA and Ritsuo NAKAMOTO
Faculty of Engineering, Ibaraki University, Hitachi
and Tennoji Senior High School, Osaka

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

1. Introduction. In this paper we shall discuss the tensor products of bounded linear operators on a complex Hilbert space H.

Following after Halmos [2], we define the numerical radius $\|T\|_N$ and the numerical range $W(T)$ as follows:

$$\|T\|_N=\sup|W(T)|$$
$$W(T)=\{(Tx,x);\|x\|=1\}.$$

Definition 1. An operator T is said to be *normaloid* if

$$\|T\|=r(T),$$

where $r(T)$ means the spectral radius of T, or equivalently

$$\|T^n\|=\|T\|^n \ (n=1,2,\cdots\cdots).$$

Definition 2. An operator T is said to be *spectraloid* if

$$\|T\|_N=r(T),$$

or equivalently

$$\|T^n\|_N=\|T\|_N^n \ (n=1,2,\cdots\cdots) \ ([4]).$$

Definition 3. An operator T is said to be *convexoid* if

$$\overline{W(T)}=\mathrm{co}\,\sigma(T),$$

where the bar denotes the closure and $\mathrm{co}\,\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of (T).

It is known that the classes of normaloids and convexoids are both contained in the class of spectraloids ([2]).

In recent years several authors paid attention to the spectral properties of the tensor products of operators on H; Brown and Pearcy [1] established

Theorem A. *If $\sigma(T)$ and $\sigma(S)$ are spectra of operators T and S respectively, then*

$$\sigma(T\otimes S)=\sigma(T)\cdot\sigma(S).$$

In connection with Theorem A, T. Saitô also proved analogous theorems among the numerical ranges of T, S and $T\otimes S$ as follows.

Theorem B ([5]).

(i) *For arbitrary operators T and S on a Hilbert space H, then*

$$\overline{W(T\otimes S)}\supseteq\overline{\mathrm{co}}\,(W(T)\cdot W(S))$$

where $\overline{\mathrm{co}}\,Z$ means the closure of convex hull of the set Z.

(ii) *Let T and S be operators on a Hilbert space H, then the condition that $T\otimes S$ is convexoid implies*

$$\overline{W(T\otimes S)}=\overline{\mathrm{co}}(W(T)\cdot W(S)).$$

Theorem C ([5]).

$T\otimes S$ is not always convexoid even if T and S are both convexoid.

V. Istrătescu and I. Istrăţescu showed

Theorem D ([3]). *If T and S are both normaloid, then the product $T\otimes S$ is also normaloid.*

Two proofs for this theorem are given in [3], one of which is based on Theorem A and the other depends on the following theorem respectively.

Theorem E ([3]). *For arbitrary operators T and S, then the approximate* proper values satisfy*

$$\prod_{ap}(T)\prod_{ap}(S)\subset\prod_{ap}(T\otimes S).$$

Motivated by Theorem C we may naturally come to mind the following question:

when does the relation $\overline{W(T\otimes S)}=\mathrm{co}\,\sigma(T\otimes S)$ hold for convexoid operators T and S?

The purpose of this paper is to give an answer to this question. Besides we shall give an alternative simplified proof of Theorem D and discuss related topics.

At the conclusion of this section we should like to express here our cordial thanks to Professor M. Nakamura and T. Saitô who encouraged us to prepare this paper.

2. Normaloid and spectraloid. In this section we shall begin to give simplified proof of Theorem D which only appeals the fact that the usual operator norm $\|T\|$ is a cross-norm in the sense of Schatten ([6]).

Theorem 1. *If T and S are normaloid, then $T\otimes S$ is also normaloid.*

Proof. As T and S are normaloid, the assertion easily follows from the relation

$$\|(T\otimes S)^n\|=\|T^n\otimes S^n\|=\|T^n\|\cdot\|S^n\|=(\|T\|\cdot\|S\|)^n=\|T\otimes S\|^n$$
$$(n=1,2,\cdots\cdots).$$

We shall give a converse of Theorem 1:

Theorem 2. *If $T\otimes S$ is non-zero normaloid, then T and S are also normaloid.*

To prove Theorem 2 we need the following lemma.

Lemma 1. *For arbitrary operators T and S, we have*

$$r(T\otimes S)=r(T)\cdot r(S),$$

that is, the spectral radius is not norm but has the cross-norm property of Schatten.

---

*) A scalar μ is said to be an *approximate proper value* for the operator T in case there exists a sequence of unit vectors x_n such that $\|Tx_n-\mu x_n\|\to 0$.

This lemma is obvious by Theorem A, but we give here a proof avoiding Theorem A.

Proof of Lemma 1.

$$r(T\otimes S)=\lim_{n\to\infty}\|(T\otimes S)^n\|^{\frac{1}{n}}=\lim_{n\to\infty}\|T^n\otimes S^n\|^{\frac{1}{n}}=\lim_{n\to\infty}(\|T^n\|^{\frac{1}{n}}\cdot\|S^n\|^{\frac{1}{n}})$$
$$=r(T)\cdot r(S).$$

Proof of Theorem 2. By Lemma 1 and the relation $r(T)\leqq\|T\|$ $r(S)\leqq\|S\|$, we have

$$r(T\otimes S)=r(T)\cdot r(S)\leqq\|T\|\cdot\|S\|=\|T\otimes S\|.$$

On the other hand $\|T\otimes S\|=r(T\otimes S)\neq 0$ by assumption, so that we conclude

$$\|T\|=r(T)\quad\text{and}\quad\|S\|=r(S).$$

By Theorems 1 and 2 we have

Corollary 1. *If T and S are not zero operator, then the following conditions are equivalent*

(i) *T and S are both normaloid,*

(ii) *$T\otimes S$ is normaloid.*

In Theorems 1 and 2 it is essential to appeal the fact that the usual operator norm $\|T\|$ has the cross-norm property in the sense of Schatten. The numerical radius $\|T\|_N$ is not a cross-norm, but we shall give analogous theorems associated with $\|T\|_N$ as follows.

Theorem 3.

(i) *If T and S are both spectraloid which satisfy the following condition*

$$(*)\qquad \|T\otimes S\|_N=\|T\|_N\cdot\|S\|_N,$$

then $T\otimes S$ is also spectraloid.

(ii) *This condition $(*)$ cannot be removed in general.*

Proof. The first assertion (i) follows from the relation

$$r(T\otimes S)=r(T)\cdot r(S)=\|T\|_N\cdot\|S\|_N=\|T\otimes S\|_N$$

by assumption $\|T\|_N=r(T)$, $\|S\|_N=r(S)$ and $(*)$.

To prove the second assertion here we give an example in which we cannot delete the condition $(*)$ in (i) of Theorem 3. Let

$$T=S=\begin{Bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{Bmatrix}$$

be a matrix 3×3 on a 3-dimensional Euclidean space E^3. By simple calculation we have $\|T\|_N=r(T)=1/2$ so that T is spectraloid. We consider an unit vector

$$x=\left(0,0,0;0,\frac{1}{\sqrt{2}},0;0,0,\frac{1}{\sqrt{2}}\right)$$

on the cross-space $E^3\otimes E^3$, we have $((T\otimes T)x, x)=\frac{1}{2}\varepsilon W(T\otimes T)$. Hence we get

$$\|T\otimes T\|_N \geqq \frac{1}{2} > \frac{1}{4} = \|T\|_N \cdot \|T\|_N.$$

On the other hand

$$r(T\otimes T) = r(T)\cdot r(T) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Thus $T\otimes T$ is not spectraloid even if T is spectraloid and the proof is complete.

We shall give a converse of Theorem 3.

Theorem 4. *If $T\otimes S$ is non-zero spectraloid, then the relation $(*)$ holds and moreover T and S are also spectraloid.*

Proof. By (i) of Theorem B and the relation $r(T)\leqq\|T\|_N$, $r(S)\leqq\|S\|_N$ we have

$$r(T\otimes S) = r(T)\cdot r(S) \leqq \|T\|_N \cdot \|S\|_N \leqq \|T\otimes S\|_N.$$

The assumption that $T\otimes S$ is non-zero spectraloid implies

$\|T\otimes S\|_N = \|T\|_N\cdot\|S\|_N$ and $\|T\|_N = r(T)$, $\|S\|_N = r(S)$ respectively.

By Theorems 3 and 4 we conclude

Corollary 2. *If T and S are non-zero operator, then the following conditions are equivalent*

(i) *T and S are both spectraloids satisfying the condition $(*)$,*

(ii) *$T\otimes S$ is spectraloid.*

3. Numerical range. In this section we state a remark concerned to Theorems B and C. Motivated by Theorem 4 we may expect the following conjecture:

the condition $\overline{W(T\otimes S)} = \overline{\mathrm{co}}\,(W(T)\cdot W(S))$ for convexoid operators T and S assures that $T\otimes S$ is convexoid.

The answer to this conjecture is affirmative, namely;

Theorem 5. *If T and S are both convexoid satisfying the condition*

$$(**) \qquad \overline{W(T\otimes S)} = \overline{\mathrm{co}}\,(W(T)\cdot W(S)),$$

then $T\otimes S$ is also convexoid.

To prove Theorem 5 we state the following lemma.

Lemma 2. *For arbitrary sets $\boldsymbol{M}$ and $\boldsymbol{N}$ of complex numbers we have*

$$\mathrm{co}\,(\boldsymbol{L}\cdot\boldsymbol{M}) = \mathrm{co}\,(\mathrm{co}\,\boldsymbol{L}\cdot\mathrm{co}\,\boldsymbol{M}).$$

Proof. We have only to show

$$\mathrm{co}\,(\boldsymbol{L}\cdot\boldsymbol{M}) \supset \mathrm{co}\,(\mathrm{co}\,\boldsymbol{L}\cdot\mathrm{co}\,\boldsymbol{M}).$$

We take $a_1, a_2 \in \boldsymbol{L}$, $b_1, b_2 \in \boldsymbol{M}$, $0\leqq s, t\leqq 1$. We have

$$(sa_1+(1-s)a_2)(tb_1+(1-t)b_2) = s(ta_1b_1+(1-t)a_1b_2) + (1-s)(ta_2b_1+(1-t)a_2b_2) \in \mathrm{co}\,(\boldsymbol{L}\cdot\boldsymbol{M}),$$

so that

$$\mathrm{co}\,\boldsymbol{L}\cdot\mathrm{co}\,\boldsymbol{M} \subset \mathrm{co}\,(\boldsymbol{L}\cdot\boldsymbol{M}),$$

consequently we have

$$\mathrm{co}\,(\mathrm{co}\,\boldsymbol{L}\cdot\mathrm{co}\,\boldsymbol{M}) \subset \mathrm{co}\,(\boldsymbol{L}\cdot\boldsymbol{M}).$$

Proof of Theorem 5. As co $\sigma(T)$ is closed, we have the following equality by assumption and Lemma 2,

$$\begin{aligned}\mathrm{co}\,(\sigma(T\otimes S)) &= \mathrm{co}\,(\sigma(T)\cdot\sigma(S)) = \mathrm{co}\,(\mathrm{co}\,\sigma(T)\cdot \mathrm{co}\,\sigma(S)) \\ &= \overline{\mathrm{co}}\,(\overline{W(T)}\cdot\overline{W(S)}) = \overline{\mathrm{co}}\,(W(T)\cdot W(S)) = \overline{W(T\otimes S)}.\end{aligned}$$

By (ii) of Theorem B the condition that $T\otimes S$ is convexoid implies (**), but we have

Theorem 6. *The condition that $T\otimes S$ is convexoid does not always imply that T and S are also convexoid.*

Proof. We consider T and S defined by

$$T=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \qquad S=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & \varepsilon & 0\end{pmatrix}$$

where $\frac{1}{2}>\varepsilon>0$.
As T is unitary it is convexoid ([2]) and S is non-convexoid and normaloid ([2]). By simple calculation $\sigma(T)=\{1,\omega,\omega^2\}$, $\omega=(-1+\sqrt{3}\cdot i)/2$, $\sigma(S)=\{0,1\}$. Following after Halmos [2] $\overline{W(T)}$ is the interior and boundary of the equilateral triangle whose vertices are $\sigma(T)=\{1,\omega,\omega^2\}$. $\overline{W(S)}$ is the union of all the closed segments that join the one point 1 to points of the closed disc with centre 0 and radius $\frac{1}{2}\varepsilon$.

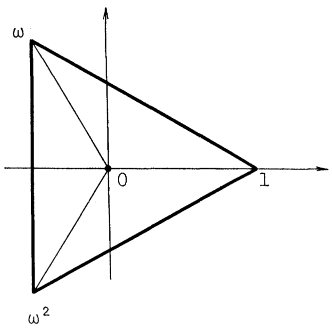
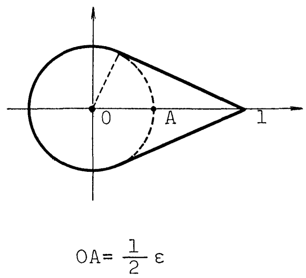


Fig. 1. $\overline{W(T)}$



$OA=\frac{1}{2}\varepsilon$

Fig. 2. $\overline{W(S)}$

$$\mathrm{co}\,(\sigma(T\otimes S))=\mathrm{co}\,(\sigma(T)\cdot\sigma(S))=\mathrm{co}\,\sigma(T),$$

on the other hand

$$|\overline{W(T\otimes M_\varepsilon)}|\leqq\|T\otimes M_\varepsilon\|=\|T\|\cdot\|M_\varepsilon\|=\varepsilon<\frac{1}{2} \qquad \text{where} \quad M_\varepsilon=\begin{bmatrix}0 & 0\\ \varepsilon & 0\end{bmatrix}$$

consequently

$$\overline{W(T\otimes M_\varepsilon)}\subseteq\overline{W(T)}$$

therefore

$$\begin{aligned}\overline{W(T\otimes S)} &= \overline{\mathrm{co}}\,(\overline{W(T)}\cup\overline{W(T\otimes M_\varepsilon)})=\overline{\mathrm{co}}\,\overline{W(T)}=\overline{W(T)}=\mathrm{co}\,\sigma(T)\\ &=\mathrm{co}\,(\sigma(T\otimes S)).\end{aligned}$$

Thus S is non-convexoid even if $T\otimes S$ is convexoid and T is unitary.

By Theorem B and Theorem 5 we can conclude

Corollary 3. *If T and S are both convexoid, then the following conditions are equivalent*

(i) $$\overline{W(T\otimes S)}=\overline{\mathrm{co}}\,(W(T)\cdot W(S)),$$

(ii) $T\otimes S$ is convexoid.

References

[1] A. Brown and C. Pearcy: Spectra of tensor products of operators. Proc. Amer. Math., Soc., **17**, 162–166 (1966).

[2] P. R. Halmos: Hilbert Space Problem Book. The University Series in Higher Mathematics, Van Nostrand (1967).

[3] V. Istrăţescu and I. Istrăţescu: On normaloid operators. Math. Zeitschr., **105**, 153–156 (1968).

[4] T. Furuta and Z. Takeda: A characterization of spectraloid operators and its generalization. Proc. Japan Acad., **43**, 599–604 (1967).

[5] T. Saitô: Numerical ranges of tensor products of operators. Tôhoku Math. Journ., **19**, 98–100 (1967).

[6] R. Schatten: A Theory of Cross-Space. Princeton University Press (1950).