146. *Free and Semi-free Differentiable Actions on Homotopy Spheres*

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1. Introduction. The theorem of Browder-Novikov enables us to construct free differentiable actions of S^1 and S^3 on homotopy spheres (see Hsiang [8]). As is shown in § 2 of this paper, every free differentiable action is obtained in such a way. Hence if we know J-groups of complex projective spaces CP^n and quaternionic projective spaces QP^n, we can classify free differentiable actions of S^1 and S^3 on homotopy spheres. In [12], Prof. S. Sasao has determined J-groups of spaces which are like projective planes. Consequently we can determine the homotopy 11-spheres admitting free differentiable S^3-actions. Let Σ_M^{11} be the generator of Θ_{11} due to Milnor. Then we shall have

Theorem 1. *Every homotopy sphere Σ which is diffeomorphic to $32k\,\Sigma_M^{11}$ for some $k\equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 15 \pmod{31}$ admits infinitely many topologically distinct S^3-actions and the remains of homotopy 11-spheres do not admit any free differentiable S^3-actions.*

Let Θ_n be the group of homotopy n-spheres and $\Theta_n(\partial\pi)$ be the subgroup consisting of those homotopy spheres which bound parallelizable manifolds. Let $\beta: \Theta_n \to \Theta_n(\partial\pi)$ be the splitting due to Brumfiel [5] and let Σ_M^{15} be the generator of $\Theta_{15}(\partial\pi)$ due to Milnor. Then we shall have

Theorem 2. *There exist at least 35 homotopy 15-spheres $\{\Sigma_k\}$ all of which admit infinitely many topologically distinct S^3-actions such that $\beta(\Sigma_k)=2^6\cdot k\Sigma_M^{15}$ where $k\equiv 0, \pm 6, \pm 8, \pm 13, \pm 14, \pm 15, \pm 17, \pm 23, \pm 26, \pm 34, \pm 35, \pm 45, \pm 48, \pm 50, \pm 51, \pm 53, \pm 55, \pm 57$ (mod. 127).*

On the other hand we shall have

Theorem 3. *A homotopy 15-sphere Σ admits no free differentiable S^3-actions if $k\not\equiv 4 \pmod 4$ where k is an integer defined by $\beta(\Sigma)=k\Sigma_M^{15}$.*

As for free S^1-actions on homotopy 15-spheres, we shall have

Theorem 4. *There exist at least 70 homotopy 15-spheres $\{\Sigma_i^{15}\}$ all of which admit infinitely many topologically distinct S^1-actions.*

An action (M^m, φ, G) is called semi-free if it is free off of the fixed

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point set, i.e. there are two types of orbits, fixed points and G. Concerning semi-free differentiable actions, Browder has studied in [4]. We shall study the situation where (Σ^m, φ, S^1) is a semi-free differentiable action of S^1 on a homotopy sphere Σ^m, and the fixed point set F^q is a homotopy sphere. Let η be a complex k-plane bundle over a homotopy q-sphere F^q. Let $\pi: P \to F$ be the associated CP^{k-1} bundle to η, and suppose $h: P \to S^q \times CP^{k-1}$ is an orientation preserving diffeomorphism such that $h^*(y)=x$, where $y=p_1^*(c_1)$, $p_1: S^q \times CP^{k-1} \to CP^{k-1}$, c_1 is the first Chern class of the canonical bundle over CP^{k-1} and x is the first Chern class of the canonical line bundle over P. Then he has proved

Theorem of Browder. *There is a semi-free S^1-action (Σ^m, φ, S^1) with fixed point set F embedded in Σ^m with (complex) normal bundle η, and such that the orbit space is $C_\pi \bigcup_h D^{q+1} \times CP^{k-1}$, where C_π is the mapping cylinder of π, and $\bigcup_h$ means we identify $P \subset C_\pi$ with $S^q \times CP^{k-1} \subset D^{q+1} \times CP^{k-1}$ via the diffeomorphism h. Every semi-free action of S^1 on a homotopy sphere of dimention >6 with fixed point set a homotopy sphere is given this way.*

He says the fixed point set F^q is untwisted when η is the trivial complex k-plane bundle and constructs semi-free differentiable S^1-actions on homotopy spheres Σ with some $F \in \Theta_q(\partial\pi)$ as untwisted fixed point set. He has posed the following problem.

"What are the homotopy spheres which are being operated on in our constructions?"

In § 3 we shall partially answer the problem. Precisely, we shall have the following

Theorem 5. *If a homotopy sphere Σ^{p+2q} admits a semi-free S^1-action with $F^p \in \Theta_p(\partial\pi)$ as untwisted fixed point set for $p \geqq 2q$, $q \equiv 3$ (mod 4) or $p=0$, then Σ^{p+2q} belongs to the inertia group $I(S^p \times CP^q)$.*

This theorem is a generalization of H. Maehara [10].

Remark. It is an interesting phenomenon that possible homotopy spheres as fixed point set are related with the inertia group $I(S^p \times CP^{q-1})$ (see Browder [4]) and possible homotopy spheres operated on are related with the inertia group $I(S^p \times CP^q)$.

Corollary. *Let Σ^{8s+1}, Σ^{8s+2} $(s \geqq 1)$ be the homotopy spheres not bounding spin-manifolds constructed by Milnor* [11] *and Anderson, Brown and Peterson* [1]. *Then Σ^{8s+1} (resp. Σ^{8s+2}) does not admit such a semi-free differentiable S^1-action as Theorem* 5.

Detailed proof will appear elsewhere.

2. Preliminary lemmas and homotopy 11-spheres admitting free S^3-actions. It is well known that to study free differentiable actions of S^3 on homotopy spheres is to study manifolds homotopically equivalent to quaternionic projective space QP^n. Let $\tau \in \widetilde{KO}(QP^n)$ (resp. $\nu \in \widetilde{KO}(QP^n)$) be the stable tangent (resp. normal) bundle of QP^n. Let $J: \widetilde{KO}(QP^n) \to J(QP^n)$ be Atiyah's fibre homotopy equivalence functor [2].

Lemma 2.1. *Given a manifold $\widetilde{QP}^n$ of the same homotopy type of QP^n. Let $g: QP^n \to \widetilde{QP}^n$ be a homotopy equivalence. Then we have*

(1) $g!\,\nu(\widetilde{QP}^n) - \nu(QP^n) \in \text{Ker } J$

(2) $\langle L(P(g!\,\tau(\widetilde{QP}^n))), \mu \rangle = \text{Index of } QP^n$,

where μ denotes the fundamental homology class of QP^n.

Proof. The former is the theorem of Atiyah [2] and the latter is proved as follows. According to Hirzebruch Index Theorem [7], we have

$$\langle L(P(\tau(\widetilde{QP}^n))), \tilde{\mu} \rangle = \text{Index of } \widetilde{QP}^n$$

where $\tilde{\mu}$ denotes the fundamental homology class of $\widetilde{QP}^n$ defined by $\tilde{\mu} = g_* \mu$. Hence we have

$$\begin{aligned} \langle L(P(g!\,\tau(\widetilde{QP}^n))), \mu \rangle &= \langle g^* L(P(\tau(\widetilde{QP}^n))), \mu \rangle \\ &= \langle L(P(\tau(\widetilde{QP}^n))), g_* \mu \rangle = \langle L(P(\tau(\widetilde{QP}^n))), \tilde{\mu} \rangle \\ &= \text{Index of } \widetilde{QP}^n = \text{Index of } QP^n. \end{aligned}$$

This completes the proof of Lemma 2.1.

Conversely we shall have

Lemma 2.2. *Given an element $\xi \in \widetilde{KO}(QP^n)$ satisfying the following conditions* (1) $\xi \in \text{Ker } J$ (2) $\langle L(P(\tau(QP^n) \oplus \xi)), \mu \rangle = \text{Index of}$ QP^n, *then we have a homotopy equivalence f of a smooth manifold $\widetilde{QP}^n$ with QP^n, $f: \widetilde{QP}^n \to QP^n$ such that $f!\,(\nu(QP^n) \oplus \xi^{-1})$ is the stable normal bundle $\nu(\widetilde{QP}^n)$ of $\widetilde{QP}^n$.*

Proof. Denote by $T(\eta)$ the Thom complex of a bundle η. Since $\xi \in \text{Ker } J$, $T(\nu(QP^n) \oplus \xi^{-1})$ is homotopy equivalent to $T(\nu(QP^n))$, i.e., the top homology class of $T(\nu(QP^n) \oplus \xi^{-1})$ is spherical. The condition (2) is that the Hirzebruch Index Theorem holds [7]. It follows from the theorem of Novikov-Browder that there exist a manifold $\widetilde{QP}^n$ and a homotopy equivalence $f: \widetilde{QP}^n \to QP^n$ such that $f!\,(\nu(QP^n) \oplus \xi^{-1})$ is the stable normal bundle $\nu(\widetilde{QP}^n)$ of $\widetilde{QP}^n$, completing the proof of Lemma 2.2.

Combining Lemmas 2.1 and 2.2, we obtain the following

Proposition 2.3. *Every manifold $\widetilde{QP}^n$ of the same homotopy type of the quoternionic projective space QP^n is obtained in the manner of Lemma* 2.2.

An outline of the proof of Theorem 1. In [12], S. Sasao has proved that $J(QP^2) = Z_{1440} \oplus Z_4$ and he has kindly informed me that $\text{Ker } J = \{24k_1$

$\oplus(-4k_1+240k_2) \mid k_1, k_2 \in Z\}$ and that the Pontrjagin classes of $\xi=24k_1 \oplus(-4k_1+240_2)$ are $P_1(\xi)=2\cdot 24k_1u$ and $P_2(\xi)=\{24k_1(2\cdot 24k_1-1)+6(-4k_1+240k_2)\}u^2$ where u denotes a generator of $H^4(QP^2)$. The Hirzebruch Index Theorem $\langle L(\tau(QP_2)\oplus\xi), \mu\rangle=$ Index of QP^2 implies that $k_2=\frac{-k_1}{70}(40k_1+1)$. Hence we have the following two cases.

Case 1_k. $k_1=70k$ and $k_2=-k(2800k+1)$, $k\in Z$

Case 2_k. $k_1=10(7k-1)$ and $k_2=-(7k-1)(400k-57)$, $k\in Z$.

Here Z denotes the set of integers.

It follows from Lemma 2.2 that we have a manifold $\widetilde{QP}^2_{1,k}$ (resp. $\widetilde{QP}^2_{2,k}$) and a principal fibration $S^3\to\Sigma^{11}_{1,k}\to\widetilde{QP}^2_{1,k}$ (resp. $S^3\to\Sigma^{11}_{2,k}\to\widetilde{QP}^2_{2,k}$) corresponding to Case 1_k (resp. Case 2_k). Let $D^4\to\tilde{W}_{1,k}\to\widetilde{QP}^2_{1,k}$ (resp. $D^4\to\tilde{W}_{2,k}\to\widetilde{QP}^2_{2,k}$) be the associated disk bundle to $S^3\to\Sigma^{11}_{1,k}\to\widetilde{QP}^2_{1,k}$ (resp. $S^3\to\Sigma^{11}_{2,k}\to\widetilde{QP}^2_{2,k}$).

Case 1_k. We have the total Pontrjagin class $P(\tilde{W}_{1,k})=1+(4+2\cdot 24\cdot 70k)u+\{12+240\cdot 4\cdot 9k+2800\cdot 24\cdot 6\cdot 4k^2\}u^2$. Hence the Eells-Kuiper μ-invariant [6] is calculated as follows.

$$\mu=\{4P_2(\tilde{W}_{1,k})P_1(\tilde{W}_{1,k})-3P_1^3(\tilde{W}_{1,k})-24(\text{Index of } \tilde{W}_{1,k})\}/2^{11}\,3\,31 \bmod 1.$$
$$=k(1+3k)(1+6k)/31 \bmod 1.$$

Case 2_k. Similarly the Eells-Kuiper μ-invariant is calculated as follows.

$$\mu(\tilde{W}_{2,k})=(5+3k)(12-14k+6k^2)/31 \bmod 1.$$

Thus we obtain

$$\{\mu(\tilde{W}_{1,k}) \bmod 1 \mid k\in Z\}\cup\{\mu(\tilde{W}_{2,k}) \bmod 1 \mid k\in Z\}$$
$$=\left\{0, \pm\frac{1}{31}, \pm\frac{2}{31}, \pm\frac{3}{31}, \pm\frac{4}{31}, \pm\frac{5}{31}, \pm\frac{10}{31}, \pm\frac{11}{31}, \pm\frac{12}{31}, \pm\frac{14}{31}, \pm\frac{15}{31} \bmod 1\right\},$$

completing the proof of Theorem 1.

3. Semi-free differentiable S^1-actions.

In case where $p=0$, Theorem 5 is due to H. Maehara [10]. We outline the proof of Theorem 5 in case where $p\geqq 2q$ and $q\equiv 3 \pmod 4$. Suppose that a homotopy sphere Σ^{p+2q} admits a semi-free differentiable S^1-action with $F^p\in\Theta_p(\partial\pi)$ as untwisted fixed point set. Then we have an equivariant diffeomorphism $f: F^p\times S^{2q-1}\to S^p\times S^{2q-1}$ such that $F^p\times D^{2q}\bigcup_f D^{p+1}\times S^{2q-1}$ is diffeomorphic to the homotopy sphere Σ^{p+2q}. Let (S^{2q+1}, φ, S^1) be the canonical free S^1-action and let $S^{2q-1}\subset S^{2q+1}$ be the natural imbedding. Then we have an equivariant tubular neighbourhood $S^{2q-1}\times D^2\subset S^{2q+1}$, i.e., we have a free S^1-action on $S^{2q-1}\times D^2$. Clearly the map $f\times id: F^p\times S^{2q-1}\times D^2\to S^p\times S^{2q-1}\times D^2$ is equivariant, hence the map $f\times id$ induces a diffeomorphism $f\times id/\sim$: $F^p\times(CP^q-\text{Int } D^{2q})\to S^p\times(CP^q-\text{Int } D^{2q})$. It is easily seen that we can regard $f\times id/\sim|F^p\times S^{2q-1}$ as f. Thus we have the following diffeomorphism

$$(f\times id/\sim)\cup id: F^p\times(CP^q-\mathrm{Int}\,D^{2q})\bigcup_{id} F^p\times D^{2q}$$
$$\rightarrow S^p\times(CP^q-\mathrm{Int}\,D^{2q})\bigcup_{f} F^p\times D^{2q}.$$

Since $p\geqq 2q$, $q\equiv 0 \mod 3$, we can easily prove that $S^p\times(CP^q-\mathrm{Int}\,D^{2q})\bigcup_f F^p\times D^{2q}$ is diffeomorphic to $S^p\times CP^q \# \Sigma^{p+2q}$ where Σ^{p+2q} is the homotopy sphere above. On the other hand, Browder has proved in [4] that $F^p\times CP^q$ is diffeomorphic to $S^p\times CP^q$. Consequently $S^p\times CP^q\#\Sigma^{p+2q}$ is diffeomorphic to $S^p\times CP^q$, i.e., $\Sigma^{p+2q}\in I(S^p\times CP^q)$. This completes the proof of Theorem 5.

Since the homotopy spheres Σ^{8s+1}, Σ^{8s+2} do not belong to the inertia groups of spin-manifolds with $\pi_1=\{1\}$ (see Lemma 9.1 of [9]), Corollary follows from Theorem 5.

Added in proof. After the preparation of the present paper, an article of H. T. Ku and M. C. Ku was published in which Theorem 1 was independently proved.

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