143. On a Property of $p-1$

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Engineering, Nihon University, Tokyo

(Comm. by Zyoiti SUETUNA, M. J. A., Oct. 13, 1969)

Erdös [1] proved in an ingenious manner that the density of the integers having a divisor between n and $2n$ tends to zero as n tends to infinity.

The purpose of this short note is to prove that the same fact holds for the sequence $\{p-1\}$, where p denotes a prime. More precisely we shall prove the following

Theorem. *The density, with respect to the sequence of all primes, of the prime p such that $p-1$ has a divisor between n and $n \exp(h^{-1}(n) \log\log n)$ tends to zero as n tends to infinity, where $h(n)$ is an arbitrary increasing function such that $h(n)\to\infty$ and $h^{-1}(n) \log\log n\to\infty$ as $n\to\infty$.*

For the proof of the theorem we need three lemmas:

Lemma 1. *Let $\omega(m)$ be the number of all prime divisors of m. Then, if $1/2\leq a<1$, we have*

$$\sum_{\substack{n\leq m\leq n \exp(h^{-1}(n)\log\log n)\\ \omega(m)\leq \alpha \log\log n}} m^{-1}=0\{\log^{\sigma_\alpha-1} n \log\log n\},$$

where $\gamma_a=a-a\log a$.

This is a trivial modification of Lemma 7 of Hooley [2].

Lemma 2. *Let $\omega_n(m)$ be the number of all prime divisors less than n of m. Then for $n\leq \log x$ we have*

$$\sum_{p\leq x} (\omega_n(p-1)-\log\log n)^2=0(\pi(x)\log\log n),$$

where $\pi(x)$ is the number of primes not exceeding x.

Lemma 3. *If c and n are less than $\log x$, then we have*

$$\sum_{\substack{p\leq x\\ p\equiv 1 \pmod c}} \left(\omega_n\left(\frac{p-1}{c}\right)-\log\log n\right)^2=0\left(\frac{\pi(x)}{\varphi(c)}\log\log n\right),$$

where $\varphi(c)$ is the Euler function.

Above two lemmas are easy applications of the Siegel-Walfisz Theorem [3, Satz 8.3].

Proof of the theorem. As in [1] we divide the integers lying between n and $n\exp(h^{-1}(n)\log\log n)$ into two classes. Namely, in the first class we put the integers $b_1, b_2, \cdots, b_y$ having at most $\frac{2}{3}\log\log n$ prime divisors and in the second class the integers $c_1, \cdots, c_z$ having more than $\frac{2}{3}\log\log n$ prime divisors.

Now the number of primes $p\leq x$ such that $p-1$ is divisible by at most one b is less than

$$(1)\qquad \sum_{i\leq y}\sum_{\substack{p\equiv 1 (\mathrm{mod}\ b_i)\\ p\leq x}} 1 \ll \sum_{i\leq y}\frac{\pi(x)}{\varphi(b_i)} \ll \pi(x)\log\log n \sum_{i\leq y} b_i^{-1},$$

since $\varphi(b_i)\gg b_i/\log\log n$. Here the last sum is

$$(2)\qquad 0((\log n)^{r\frac{2}{3}-1}\log\log n)=0((\log n)^{-0.06}),$$

by Lemma 1 and the definition of $b's$.

Again as in [1] we arrange the primes $p\leq x$ such that $p-1$ is divisible by a c into two sets. In the first class we put those of the form $p-1=c_1k$ where k has at most $\frac{2}{3}\log\log n$ prime divisors less than n. Then the number of primes in this class is less than

$$(3)\qquad \begin{aligned} & o\left\{\frac{\pi(x)}{\log\log n}\sum_{j\leq z}\frac{1}{\varphi(c_j)}\right\} \\ &= O\left\{\frac{\pi(x)}{\log\log n}\sum_{n\leq m\leq n\exp(h^{-1}(n)\log\log n)}\varphi^{-1}(m)\right\} \\ &= o\{h^{-1}(n)\pi(x)\}, \end{aligned}$$

since Lemma 3 and the fact that there exist two constants B_1 and B_2 such that

$$\sum_{m\leq M}\varphi^{-1}(m)=B_1\log M+B_2+o\left(\frac{1}{\log M}\right).$$

Obviously for the primes in the second class we have

$$\omega_n(p-1)\geq\frac{4}{3}\log\log n,$$

and hence from Lemma 2 the number of primes in the second class is

$$(4)\qquad O\left(\frac{\pi(x)}{\log\log n}\right).$$

Therefore from (1), (2), (3) and (4) we can conclude that the desired density is

$$O(h^{-1}(n)).$$

This proves the theorem.

References

[1] P. Erdös: Note on sequence of integers no one of which is divisible by any other. J. London Math. Soc., **10**, 126–128 (1935).

[2] C. Hooley: On the representation of a number as the sum of two squers and a prime. Acta Math., **97**, 189–210 (1957).

[3] K. Pracher: Primzahlverteilung. Springer (1957).