201. A Characterization of Pseudo-open Images of M-Spaces

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P. S. Alexandroff [1] asked which spaces can be represented as images of "nice" spaces under "nice" continuous maps. Until recently, the "nice" spaces have generally consisted of metric spaces. In 1968, J. Nagata rephrased the question, "which spaces can be represented as images of M-spaces under 'nice' continuous maps." He has since characterized quotient and bi-quotient images of M-spaces in [6], and later he answered the question for open images of M-spaces [4].

Consider the following

Definition 1. A map $f: X \rightarrow Y$ is said to be pseudo-open iff, given any $y \in Y$ and U open about $f^{-1}(y)$ in X, then $y \in \operatorname{int} f(U)$.

E. Michael and Nagata, working independently, suggested essentially the following characterization:

Theorem 1. *A space Y is the pseudo-open image of an M-space X iff Y has the following property*:

$y \in \mathrm{Cl}\, B$ for $B \subseteq Y$ implies that there is a family $\{U_1, U_2, \cdots\}$ of subsets of Y such that

(1) *$y \in U_i$ for all i;*

(2) *$y \in \mathrm{Cl}(U_i \cap B)$ for all i,*

(3) *If $x_i \in U_i$ for all i, then the sequence $\{x_i\}$ has a cluster point x' in $\bigcap_{i=1}^{\infty} U_i$.*

Definition 2. A space satisfying the above property will be said to have property-(P). The Nagata-Michael conjecture is correct, as this paper will demonstrate. First, recall some information from Nagata [6].

Definition 3. A sequence $A_1 \supseteq A_2 \supseteq \cdots$ of subsets of a space X is called a q-sequence if any point sequence $\{x_i : i=1, 2, \cdots\}$ satisfying $x_i \in A_i$ has a cluster point in $\bigcap A_i$. (This is (3) of property-(P).)

Definition 4. A sequence $U_1, U_2, \cdots$ of open neighborhoods of a point x in X is called a q-sequence of neighborhoods if $U_1 \supseteq \bar{U}_2 \supseteq U_2 \cdots$ and if any point sequence $\{x_i : i=1, 2, \cdots\}$ satisfying $x_i \in U_i$ has a cluster point.

Lemma 1. *Let f be a continuous map from a space X onto a space Y. If $\{A_i : i=1, 2, \cdots\}$ is a q-sequence in X, then $\{f(A_i) : i, 2, \cdots\}$*

is a q-sequence in Y.

Lemma 2. *Any point of an M-space has a q-sequence of neighborhoods.*

With the aid of the above, we can now prove Theorem 1.

First, let $f: X \to Y$ be a pseudo-open map, where X is M. Take $B \subseteq Y$ such that $y \in \mathrm{Cl}\, B$, and let $x \in f^{-1}(y)$. Since X is an M-space, there exists a q-sequence $\{U_1, U_2, \cdots\}$ of neighborhoods of x. Further, by Lemma 1, $\{f(U_i): i=1, 2, \cdots\}$ is a q-sequence in Y, and $y \in f(U_i)$ for each i. Thus (1) and (3) of property$-(P)$ hold.

Observe that there is an $x \in f^{-1}(y)$ such that $x \in \mathrm{Cl}\, f^{-1}(B)$. (If not, $x \notin \mathrm{Cl}\, f^{-1}(B)$ implies the existence of an open neighborhood $0(y)$ of $f^{-1}(y)$ such that $0(y) \cap f^{-1}(B) = \emptyset$. Then, by the pseudo-openness of f, $y \in \mathrm{int}\, f(0(y))$ and $f(0(y)) \cap B = \emptyset$. Thus $y \notin \mathrm{Cl}\, B$. Contradiction.)

We must now only show that $y \in \mathrm{Cl}\,(f(U_i) \cap B)$. So take $x \in f^{-1}(y)$ such that $x \in \mathrm{Cl}\, f^{-1}(B)$. Then for any neighborhood of x, call it $N(x)$, $N(x) \cap f^{-1}(B) \neq \emptyset$. But $y \in \cap U_i$ implies that $N(x) = U_i \cap f^{-1}(U(y))$ is a neighborhood of x for each i and for a given neighborhood $U(y)$ of y. Then

$$U_i \cap f^{-1}(U(y)) \cap f^{-1}(B) \neq \emptyset$$

implies $$f(U_i) \cap U(y) \cap B \neq \emptyset \text{ for any } U(y).$$

Finally, $$y \in \mathrm{Cl}\,(f(U_i) \cap B).$$

Now suppose Y has property-(P). Take a q-sequence $\alpha = \{U_1, U_2, \cdots\}$ with non-empty intersection; let $C(\alpha) = \bigcap U_i$. Define Y_α by means of the neighborhood basis $\mathcal{N}(y)$ of $y \in Y$ such that

$\mathcal{N}(y) = \{U_i \cap U(y): i=1, 2, \cdots;\ U(y)$ an original neighborhood of $y \in Y\}$ if $y \in C(\alpha)$ and,

$\mathcal{N}(y) = \{y\}$ if $y \notin C(\alpha)$.

Then define an open cover

$$\mathcal{U}_{i,\alpha} = \{U_i, \{z\}: z \in Y_\alpha - U_i\}$$

Let $X = \bigcup_{\alpha \in \Omega} Y_\alpha$ be the discrete sum, with an open cover of X of the type

$$\mathcal{U}_i = \bigcup_{\alpha \in \Omega} \mathcal{U}_{i,\alpha}$$

where $\Omega =$ all q-sequences with nonempty intersection. Then X as defined is an M-space (see Nagata [6], p. 28).

Next let f be the map from X onto Y obtained by combining the identity maps

$$f_\alpha: Y_\alpha \to Y.$$

f is clearly continuous. It remains to show f pseudo-open.

Assume $z \in Y$ and U is open about $f^{-1}(z)$ in X. If $z \notin \mathrm{int}\, f(U)$, then $z \in \mathrm{Cl}(Y - f(U))$. So for every neighborhood $U(z)$ of z, $z \in \mathrm{Cl}(U(z) \cap (Y - f(U)))$. Then there exists $\alpha = \{U_1, U_2, \cdots\}$ as in property-(P) such that $z \in U_i$ for all i and such that $z \in \mathrm{Cl}(U_i \cap U(z) \cap (Y - f(U)))$.

Now $U_i \cap U(z) \cap (Y - f(U)) \neq \emptyset$ implies

$$Y_\alpha \cap f^{-1}(U_i \cap U(z) \cap (Y - f(U))) \neq \emptyset \quad \text{in} \quad Y_\alpha.$$

But

$$Y_\alpha \cap f^{-1}(U_i \cap U(z) \cap (Y - f(U))) \subseteq f_\alpha^{-1}(U_i) \cap f_\alpha^{-1}(U(z)) \cap (X - U).$$

This last is then nonnull for all i and for every neighborhood $U(z)$. But for $x = f^{-1}(z) \cap Y_\alpha$, every basic neighborhood has the form

$$f_\alpha^{-1}(U_i) \cap f_\alpha^{-1}(U(z))$$

since $z \in C(\alpha)$. Thus U isn't open. Contradiction.

Thus Theorem 1 holds.

Recalling a definition of E. Michael, we note that we have also shown:

Corollary 1. *A regular space X has property-(P) iff X is the pseudo-open image of a regular q-space.*

References

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