198. Two Spaces whose Product has Closed Projection Maps

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This note will give several equivalent properties with that of two spaces in the title and some properties of the spaces. This is also a preparation for the forthcoming paper [2].

Throughout this note, spaces are Hausdorff. We use notations in [1].

Definition 1 (cf. [4, p. 365]). A set A in $X\times Y$ is called to be *upper semi-continuous at* $a\in X$ if for any open set G in Y containing $A[a]$ there is $U\in\mathfrak{N}_a$ with $\bigcup_{x\in U} A[x]\subset G$. A is called *upper semi-continuous at* X if A is upper semi-continuous at every point of X.

It is easily seen that A is upper semi-continuous at X if and only if the set $\{x\in X; A[x]\subset G\}$ is open in X for every open G of Y.

Definition 2. Let a be a point of X. A space Y with the following property is called to be *upper compact at* a. Let Z be any subset of X with $a\in\bar{Z}$, and let $\{A_x; x\in Z\}$ be any family of non-empty subsets of Y, then $\limsup_a A_x\neq\emptyset$. Y is called *upper compact at* X when Y is upper compact at every point of X.

In this definition we can replace $\bar{Z}$ by $\bar{Z}-Z$.

The following is seen easily.

Proposition 1. *A closed subset of a space which is upper compact at* $a\in X$ *is upper compact at* a.

Proposition 2. *In order that* Y *is upper compact at* $a\in X$, *it is necessary and sufficient that for any subset* Z *of* X *with* $a\in\bar{Z}$, *and for any family* $\{B_U; U\in\mathfrak{N}_a\}$ *of subsets of* Y *such that*

$$\bigcap_{U\ni x}\overline{B_U}\neq\emptyset$$

for every point $x\in Z$, *it holds* $\bigcap_{U\in\mathfrak{N}_a}\overline{B_U}\neq\emptyset$.

Proof. *Necessity.* Put

$$A_x=\bigcap_{U\ni x}\overline{B_U}$$

for $x\in Z$, then A_x is not empty, so

$$\emptyset\neq\bigcap_{U\in\mathfrak{N}_a}\overline{\bigcup_{x\in U}A_x}\subset\bigcap_{U\in\mathfrak{N}_a}\overline{B_U}.$$

Sufficiency. Let $\{A_x; x\in Z\subset X\}$, $a\in\bar{Z}$, be an arbitrary family of non-empty subsets of X. Put

$$B_U = \bigcup_{x \in U} A_x,$$

then

$$\bigcap_{U \ni x} \overline{B_U} \supset A_x \neq \emptyset$$

for $x \in Z$, so we have

$$\bigcap_{U \in \mathfrak{N}_a} \overline{\bigcup_{x \in U} A_x} = \bigcap_{U \in \mathfrak{N}_a} \overline{B_U} \neq \emptyset.$$

Corollary. *In order that Y is upper compact at $a \in X$, it is necessary and sufficient that for any open cover $\mathfrak{G} = \{G_U\,;\, U \in \mathfrak{N}_a\}$ of Y and for any subset Z of X with $a \in \bar{Z}$, there is a point $x_0 \in Z$ such that $\{G_U\,;\, U \ni x_0\}$ is a subcover of $\mathfrak{G}$.*

Proposition 3. *The property that Y is upper compact at $a \in X$ is necessary and sufficient in order that any closed subset A of $X \times Y$ is upper semi-continuous at a.*

Proof. *Necessity.* Suppose that

(1) $$\limsup_a A_x = \emptyset$$

for some family $\{A_x\,;\, x \in Z\}$ of non-empty A_x and for some $Z \subset X$ with $a \in \bar{Z}$. Then there is $U_0 \in \mathfrak{N}_a$ such that for some non-empty open $G \subset Y$,

(2) $$A_x \not\subset G$$

for all $x \in U_0$. Take a point $y \in G$ and put

$$B = \bigcup_{x \in U_0} (x, A_x \cup \{y\}).$$

Then, for every $x \in U_0$,

$$\bar{B}[x] \supset B[x] = A_x \cup \{y\},$$
$$\bar{B}[x] \not\subset G,$$

namely, for any $U \in \mathfrak{N}_a$ there is $x \in U$ with

(3) $$\bar{B}[x] \not\subset G.$$

On the other hand, from Proposition 2 in [1] and (1) we have

$$\bar{B}[a] = \limsup_a (A_x \cup \{y\}) = \{y\} \subset G,$$

which means together with (3) that $\bar{B}$ is not upper semi-continuous at a.

Sufficiency. Suppose that there is a non-empty closed set $A \subset X \times Y$ which is not upper semi-continuous at a. (An empty set is upper semi-continuous.) There is an open set G including $A[a]$ such that for any $U \in \mathfrak{N}_a$ there is $x_U \in U$ with $A[x_U] \not\subset G$. Put $B_x = A[x] - G$ for $x \in X$, then

$$B = \bigcup_{x \in X} (x, B_x) = A - (X \times G)$$

is closed in $X \times Y$. $B[x_U] \neq \emptyset$ and $a \in \overline{\{x_U\,;\, U \in \mathfrak{N}_a\}}$. Since Y is upper compact at a, we have from Corollary 1 to Proposition 2 in [1]

$$\emptyset \neq \limsup_a B[x_U] \subset \limsup_a B[x] = B[a] = A[a] - G = \emptyset,$$

the contradiction.

Proposition 4. *Y is upper compact at X if and only if the projection map of $X \times Y$ onto X is closed.*

Proof. Suppose that Y is upper compact and A is a closed subset of $X\times Y$, and that there is a point

$$a\in\overline{\mathrm{proj}_X A}-\mathrm{proj}_X A.$$

From Corollary 1 to Proposition 2 in [1], we have $\limsup_a A[x]=A[a]=\emptyset$, which contradicts the upper compactness of Y.

Conversely, suppose that proj_X is closed. Consider any family $\{A_x; x\in Z\subset X\}$, $a\in\bar{Z}$, of non-empty $A_x\subset Y$, and put

$$A=\bigcup_{x\in Z}(x, A_x).$$

Since $Z\subset\mathrm{proj}_X \bar{A}$, we have

$$a\in\bar{Z}\subset\overline{\mathrm{proj}_X \bar{A}}=\mathrm{proj}_X \bar{A},$$
$$\limsup_a A_x=\bar{A}[a]\neq\emptyset$$

by Proposition 2 in [1].

The following is essentially well known.

Corollary 1. *A space Y is compact if and only if Y is upper compact at any space.*

Definition 3. Let $\mathfrak{m}$ be a cardinal number. A space is called $\mathfrak{m}$-*compact* if every open cover of power $\leqq\mathfrak{m}$ of the space has a finite subcover.

Corollary 2 (cf. the footnote on p. 234 of [5]). *If a point a of X has the character $\leqq\mathfrak{m}$, and if Y is $\mathfrak{m}$-compact, then Y is upper compact at a.*

Though the following is essentially known, we shall give a proof in our version.

Proposition 5. *If a non-discrete space X satisfies the first axiom of countability, then Y is upper compact at X if and only if Y is countably compact.*

Proof. From Corollary 2 above, it suffices to verify "only if" part. Suppose that a countable open cover $\mathfrak{G}=\{G_1, G_2, \cdots\}$ of Y is given. Take a non-isolated point a in X, then we can select a sequence $\{x_1, x_2, \cdots\}$ of points of X which converges to a and a neighborhood base $\{U_1, U_2, \cdots\}$ of a such that $x_i\notin U_n$ for all $i<n$. Considering $G_{U_i}=G_i$ and $Z=\{x_1, x_2, \cdots\}$, and applying Corollary to Proposition 2, we have a finite subcover of $\mathfrak{G}$.

Example. Let ω_1 be the first uncountable ordinal number, and denote by $W(\alpha)$ for an ordinal number α the space consisting of all ordinals less than α with the order topology.

(1) By Proposition 5, $W(\omega_1)$ is upper compact at itself.

(2) From the definition, $W(\omega_1)$ is not upper compact at $W(\omega_1+1)$, i.e., not upper compact at ω_1.

Definition 4. Let $\mathfrak{m}$ be a cardinal number. A space X is said to be $\mathfrak{m}$-*paracompact* if any open cover with power $\leqq\mathfrak{m}$ of X admits a locally finite open refinement. Let $\mathfrak{n}$ be a cardinal number. A space X is said to be $\mathfrak{n}$-*Lindelöf* if any open cover of X includes a subcover of power $\leqq \mathfrak{n}$.

Definition 5. A family $\{G_\lambda ; \lambda \in \Lambda\}$ of open sets in a space is said an *open base for closed sets* if for any closed set A and any open set E containing A there is $\lambda \in \Lambda$ with $A \subset G_\lambda \subset E$.

Proposition 6. *A space X is compact and metrizable if and only if it is regular and has an open base of power $\leqq \aleph_0$ for closed sets.*

Proof. Suppose that X is compact and metrizable, then it has a countable open base $\{E_n ; n=1, 2, \cdots\}$. Denote by Γ the totality of all the finite sets of natural numbers, and put

$$G_\gamma = \bigcup_{n \in \gamma} E_n$$

for $\gamma \in \Gamma$, then $\{G_\gamma ; \gamma \in \Gamma\}$ is an open base for closed sets in X with $\|\Gamma\| \leqq \aleph_0$, where $\|\Gamma\|$ is the power of Γ.

Conversely, suppose that a regular space X has an open base $\{G_n ; n=1, 2, \cdots\}$ for closed sets. Since it is an open base, we can consider that X is a metric space with a distance function d. If X is not compact, then there is a sequence $\{x_n ; n=1, 2, \cdots\}$ of points without accumulation point and a sequence $\{r_n ; n=1, 2, \cdots\}$ of positive numbers such that $U_n = \{x ; d(x_n, x) < r_n\}$ does not include any x_i with $i \neq n$. For any set α of natural numbers there is $G_{n(\alpha)}$ such that

$$\{x_i ; i \in \alpha\} \subset G_{n(\alpha)} \subset \bigcup_{i \in \alpha} U_i,$$

and $G_{n(\alpha)} \neq G_{n(\alpha')}$ for $\alpha \neq \alpha'$, which is impossible because of $2^{\aleph_0} > \aleph_0$.

Since an open base for closed sets is an open base for the space, we easily have

Proposition 7. *If a space has an open base for closed sets of power $\leqq \mathfrak{m}$, then it is $\mathfrak{m}$-Lindelöf.*

Definition 6. Let $\mathfrak{m}$ be a cardinal number, and A a subset of X. A point $a \in \bar{A}$ is said to be an $\mathfrak{m}$-*point of* A if for any family $\mathfrak{F} = \{U\}$ of neighborhoods of a with power $\leqq \mathfrak{m}$, it holds

$$A \cap (\bigcap_{U \in \mathfrak{F}} U) \neq \emptyset. \tag{1}$$

If (1) holds for any A with $a \in \bar{A}$, then a is called an $\mathfrak{m}$-*point.*

In this definition we can replace $\bar{A}$ by $\bar{A} - A$. A P-point in the sense of [3] is an $\aleph_0$-point in our sense.

Proposition 8. *An $\mathfrak{m}$-Lindelöf space Y is upper compact at an $\mathfrak{m}$-point $a \in X$.*

Proof. Suppose that $\{A_x \subset Y ; x \in Z \subset X\}$, $a \in \bar{Z}$, is given with

$$\bigcap_{U \in \mathfrak{N}_a} \overline{\bigcup_{x \in U} A_x} = \emptyset.$$

$\{C(\overline{\bigcup_{x \in U} A_x}) ; U \in \mathfrak{N}_a\}$ is an open cover of Y, so there is a subfamily $\mathfrak{F}$ of $\mathfrak{N}_a$ with power $\leqq \mathfrak{m}$ such that $\{C(\overline{\bigcup_{x \in U} A_x}) ; U \in \mathfrak{F}\}$ is a cover of Y. Since

a is an m-point, there is a point $z \in Z \cap \{\bigcap_{U \in \mathfrak{F}} U\}$, and

$$\emptyset \neq A_z \subset \bigcap_{U \in \mathfrak{F}} \overline{\bigcup_{x \in U} A_x} = \emptyset,$$

the contradiction.

References

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