190. *Distributions as the Boundary Values of Analytic Functions*

By Richard D. CARMICHAEL
Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, Virginia, U. S. A.

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The aim of this note is to announce some theorems (Theorems 1–8) concerning distributions as the boundary values of functions which are analytic in a subset of $\boldsymbol{C}^n$.

The n dimensional notation used here will be that of Schwartz [1]. $C\subseteq \boldsymbol{R}^n$ is a cone with vertex at zero if $y\in C$ implies $\lambda y\in C$ for all positive scalars λ. The intersection of C with the unit sphere $|y|=1$ is called the projection of C and is denoted pr C. Let C' be a cone such that $\operatorname{pr}\bar{C}'\subset \operatorname{pr} C$; then C' will be called a compact subcone of C. The function

$$u_C(t)=\sup_{y\in \operatorname{pr}C}(-\langle t, y\rangle)$$

is the indicatrix of the cone C. $0(C)$ will denote the convex envelope of C. $T^C=\boldsymbol{R}^n+iC$, where C is an open connected cone, is a tubular radial domain. The Fourier transform of $f(t)\in L^1$ will be denoted by $\hat{f}$ or $\mathcal{F}[f(t);x]$ and is defined as

$$\hat{f}(x)=\int_{\boldsymbol{R}^n} f(t)e^{2\pi i\langle x,t\rangle}dt.$$

We refer to Schwartz [1] and Gel'fand and Shilov [2] for definitions and facts concerning the distribution spaces.

1. Distribution boundary values in Z'. Lauwerier [3] has shown that functions which are analytic in $\operatorname{Im}(z)>0$, $z\in \boldsymbol{C}^1$, and which are bounded by a polynomial have distributional boundary values in the Z' topology. We extend the results of Lauwerier to functions which are analytic in tubular radial domains, T^C.

Theorem 1. *Let $f(z)$ be analytic in T^C. For any arbitrary compact subcone C' of C let $f(z)$ satisfy*

$$|f(z)|\leq K(C')(1+|z|)^N e^{2\pi(A+\sigma)|y|},\ z\in T^{C'}, \tag{1}$$

for all $\sigma>0$, where A is a nonnegative real number, N is any real number, and $K(C')$ is a constant depending on C'. Then $f(z)$ has a distributional boundary value $U\in Z'$ which is the Fourier transform of an element $V\in \mathcal{D}'$ which vanishes for $u_C(t)>A$.

Let P be a constant such that $N-2P\leq -n-\varepsilon$ for all $\varepsilon>0$; and let $B\in \boldsymbol{R}^1$, $B>0$, be such that

$$Q(1+|z|)^M \geq |B+\langle z, z\rangle| \geq (1+|z|)^2, \; z \in \boldsymbol{R}^n + iC'',$$

where C'' is an arbitrary compact subset of C and Q and M are constants. It follows that

$$(2) \qquad g(t) = \int_{\boldsymbol{R}^n} f(z)(B+\langle z, z\rangle)^{-P} e^{-2\pi i\langle z, t\rangle} dx$$

is independent of $y \in C$ and vanishes if $u_C(t) > A$. It is immediate from (2) that $e^{-2\pi\langle y, t\rangle} g(t) \in L^2$ and $f(z)(B+\langle z, z\rangle)^{-P} = \mathcal{F}[e^{-2\pi\langle y, t\rangle} g(t)\,; x]$ in the L^2 sense. Letting $\Psi \in Z$ and $\Phi \in \mathcal{D}$ such that $\hat{\Phi} = (B+\langle z, z\rangle)^P \Psi$ we have

$$\langle f(z), \Psi\rangle = \langle e^{-2\pi\langle y, t\rangle} g(t), \Phi\rangle, \; z \in T^C.$$

Then as $y \to 0$, $y \in C' \subset C$ we see that

$$\langle e^{-2\pi\langle y, t\rangle} g(t), \Phi\rangle \to \langle g(t), \Phi\rangle = \langle U, \Psi\rangle$$

where $U \in Z'$ is the Fourier transform of $V = (B - \Delta)^P g(t) \in \mathcal{D}'$, $\Delta = \frac{1}{4\pi^2} \sum_{j=1}^n \frac{\partial^2}{\partial t_j^2}$. This proves Theorem 1. Note that $g(t)$ is continuous and bounded as $0(e^{2\pi\langle y, t\rangle})$ for all $y \in C$.

We define $\check{U}$ by $\langle \check{U}, \Phi\rangle = \langle U, \Phi(-t)\rangle$ and denote a neighborhood of the origin with radius R by $N(0, R)$. As a converse result to Theorem 1 we obtain

Theorem 2. *Let $U = D^\alpha g(t)$, where $g(t)$ is continuous and bounded as $0(e^{2\pi\langle B, t\rangle})$ for all $B \in 0(C)$. Let U vanish if $u_C(t) > 0$. Then there exists a function $f(z)$ which is analytic in $T^{0(C)}$, and for any compact subcone C' of C we have*

$$(3) \qquad |f(z)| \leq K(C')(1+|z|)^N, \; z \in T^{C' \setminus (C' \cap N(0, R))},$$

where R is fixed. Furthermore $f(z) \to \mathcal{F}(\check{U}) \in Z'$ in the topology of Z' as $y \to 0$, $y \in C'$.

The desired function is $f(z) = \langle U, e^{2\pi i\langle z, t\rangle}\rangle$, which can be shown to be analytic in $T^{0(C)}$. The boundedness condition follows by a straightforward calculation. Let $\Psi \in Z$ and $\Psi = \hat{\Phi}$, $\Phi \in \mathcal{D}$. We obtain

$$\langle\langle U, e^{2\pi i\langle z, t\rangle}\rangle, \Psi\rangle = \langle U, e^{-2\pi\langle y, t\rangle} \Phi(-t)\rangle.$$

As $y \to 0$ in $C' \subset C$, it follows that

$$\langle\langle U, e^{2\pi i\langle z, t\rangle}\rangle, \Psi\rangle \to \langle \check{U}, \Phi\rangle = \langle V, \Psi\rangle$$

where $V = \mathcal{F}(\check{U})$.

The results of this section suggest a solution to the Hilbert problem for Z'. Let $U \in Z'$ be such that $\mathcal{F}^{-1} U = \check{V} \in \mathcal{D}'$, $V = D^\alpha g(t)$, where $g(t)$ is continuous and bounded as $0(e^{2\pi\langle B, (|t_1|, \cdots, |t_n|)\rangle})$ for all B, B being an n-tuple of positive real numbers. Denote $G_\delta = \{z : \delta_j \operatorname{Im}(z_j) > 0, \delta = (\delta_1, \cdots, \delta_n), \delta_j = \pm 1, j = 1, \cdots, n\}$. Denote $\mathcal{O}_\delta = \{t : \delta_j t_j > 0, j = 1, \cdots, n\}$; and let

$$g_{\mathcal{O}_\delta}(t) = -\begin{cases} g(t), & t \in \mathcal{O}_\delta \\ 0, & t \notin \mathcal{O}_\delta \text{ and } t \notin \text{boundary of } \mathcal{O}_\delta. \end{cases}$$

Put $V_{\mathcal{O}_\delta} = D^\alpha g_{\mathcal{O}_\delta}(t)$. Then $\langle V_{\mathcal{O}_\delta}, e^{2\pi i\langle z, t\rangle}\rangle \to \mathcal{F}(\check{V}_{\mathcal{O}_\delta})$ in Z' as $\operatorname{Im}(z) \to 0$, $z \in G_\delta$. We obtain $U = \sum_\delta \mathcal{F}(V_{\mathcal{O}_\delta})$ where there are 2^n elements in this sum.

2. Distributional boundary values in $\mathcal{S}'$. We obtain a boundary value theorem for $\mathcal{S}'$ under the assumption of the boundedness condition (1). Such theorems have relevance in quantum field theory (see [4]).

Theorem 3. *Let $f(z)$ be analytic in T^C and satisfy* (1). *Let $f(z) \to U$ in the topology of $\mathcal{S}'$ as $y \to 0$, $y \in C'$, C' being an arbitrary compact subcone of C. Then $U \in \mathcal{S}'$; there exists an element $V \in \mathcal{S}'$ which vanishes if $u_C(t) > A$; and $f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^C$, as elements of $\mathcal{S}'$.*

The fact that $U \in \mathcal{S}'$ follows immediately from the completeness of the $\mathcal{S}'$ topology. The element V is $(B-\Delta)^P g(t)$ which is shown to be in $\mathcal{S}'$ using Theorem 1. The equality $f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^C$, in $\mathcal{S}'$ follows using the relation $f(z)(B+\langle z, z\rangle)^{-P} = \mathcal{F}[e^{-2\pi\langle y,t\rangle}g(t); x]$ in the L^2 sense which was obtained in the proof of Theorem 1. We note this result has been obtained by DeJager [5] and Beltrami and Wohlers ([6], [7]) if C is $\mathrm{Im}\,(z) > 0$, $z \in \boldsymbol{C}^1$ and by L. Gårding ([4], p. 61) if C is the forward or backward light cone. We note that Vladimirov [8, p. 235] has characterized functions which are analytic in tubular cones and which have distributional boundary values in $\mathcal{S}'$ using a boundedness condition which is more restrictive than (1).

Let $f(z)$ be bounded in $\mathcal{S}'$ as a function of x for any $y \in C$. Since $\mathcal{S}'$ is a Montel space then $f(z)$ converges to some $U \in \mathcal{S}'$ as $y \to 0$, $y \in C'$, C' being any arbitrary compact subcone of C. Using Theorem 3 we have proved

Theorem 4. *Let $f(z)$ be analytic in T^C and satisfy* (1). *Let $f(z)$ be bounded in $\mathcal{S}'$ as a function of x for any $y \in C$. Then there exists an element $U \in \mathcal{S}'$ such that $f(z) \to U$ in $\mathcal{S}'$ as $y \to 0$, $y \in C'$; and the conclusions of Theorem* 3 *hold.*

This result extends a theorem of Swartz [9].

A theorem for functions analytic in an octant G_δ which gives other conditions for the convergence in $\mathcal{S}'$ of Theorem 3 to be proved is

Theorem 5. *Let $f(z)$ be analytic in $G_{(1,\dots,1)}$ and let it be continuous on $\mathrm{Im}\,(z_j) = 0$, $j = 1, \cdots, n$. Let $f(z)$ satisfy.*

(4) $\quad |f(z)| \le Q(1+|z|)^N e^{2\pi\langle A,(|y_1|,\dots,|y_n|)\rangle}$, $\mathrm{Im}\,(z_j) \ge 0$, $j = 1, \cdots, n$,

for some constants Q and N and for any n-tuple A of real numbers. Then there exists an element $U \in \mathcal{S}'$ such that $\mathrm{supp}\,(U) \subseteq S_A = \{t;\ -A_j \le t_j < \infty,\ j = 1, \cdots, n\}$; $f(z) = \langle U, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in G_{(1,\dots,1)}$; and $f(z) \to \hat{U}$ in $\mathcal{S}'$ as $\mathrm{Im}\,(z) \to 0$.

A similar result holds for each octant G_δ.

As a converse result we obtain

Theorem 6. *Let $U \in \mathcal{S}'$ and let U vanish if $u_C(t) > A \ge 0$. Then there exists a function $f(z)$ which is analytic in $T^{0(C)}$; $f(z)$ satisfies* (3);

$f(z) \to \hat{U}$ in $\mathcal{S}'$ as $y \to 0$, $y \in C' \subset C$; and $f(z)$ is bounded in $\mathcal{S}'$ as a function of x for any $y \in C$.

The function is $f(z) = \langle U, \alpha(t) e^{2\pi i \langle z, t \rangle} \rangle$ where $\alpha(t) \in \mathcal{E}$, $\alpha(t) = 1$ on $\operatorname{supp}(U) = \{t : u_C(t) > A \geq 0\}$, and $\alpha(t)$ vanishes in a neighborhood of $\operatorname{supp}(U)$. The fact that $f(z) \to \hat{U}$ in $\mathcal{S}'$ follows using essentially the same argument as in Theorem 2 except that $\Psi \in \mathcal{S}$ and $\Phi = \hat{\Psi} \in \mathcal{S}$ and the topology here is that of $\mathcal{S}'$. To show that $f(z)$ is bounded in $\mathcal{S}'$ we show that $f(z)$ is bounded on bounded sets of $\mathcal{S}$. Let $\Phi \in B$, a bounded set of $\mathcal{S}$. Then

$$\langle f(z), \Phi \rangle = \langle U, \alpha(t) e^{-2\pi \langle y, t \rangle} \hat{\Phi} \rangle. \tag{5}$$

It follows that $\alpha(t) e^{-2\pi \langle y, t \rangle} \hat{\Phi}$ is bounded in $\mathcal{S}$ for $y \in C$. Since $U \in \mathcal{S}'$, we have by (5) that $\sup_{\Phi \in B} |\langle f(z), \Phi \rangle|$ is finite for $z \in T^C$. We note that Schwartz ([1], p. 235) has characterized bounded sets in $\mathcal{S}$.

3. Distributional boundary values in $\mathcal{D}'_{L^P}$.

Consider $\mathcal{D}'_{L^P}$ as a subset of $\mathcal{S}'$ with the topology of $\mathcal{S}'$. We generalize some results of Beltrami and Wohlers ([6], [7]) for functions analytic in an octant G_δ. For convenience we shall state the results for $z \in G_{(1,\cdots,1)}$ and note that similar theorems hold for each of the octants. Recall the definition of the set $S_{(0,\cdots,0)}$ from Theorem 5.

Theorem 7. *Let U be a distribution such that* $\operatorname{supp}(U) \subseteq S_{(0,\cdots,0)}$ *and $\hat{U} \in \mathcal{D}'_{L^P}$ for some P, $1 \leq p \leq 2$. Then $U = \sum_{|\alpha| \leq m} t^\alpha g_\alpha(t)$, where g_α is continuous and bounded if $P = 1$ or $g_\alpha \in L^q$, $1/p + 1/q = 1$, $1 < p \leq 2$;*

$$f(z) = \langle U, e^{2\pi i \langle z, t \rangle} \rangle = \frac{1}{(2\pi i)^n} \left\langle \hat{U}, \prod_{j=1}^{n} \frac{1}{t_j - z_j} \right\rangle, \; z \in G_{(1,\cdots,1)},$$

as elements of $\mathcal{S}'$; and $f(z) \to \hat{U} \in \mathcal{D}'_{L^P}$ in the topology of $\mathcal{S}'$ as $\operatorname{Im}(z) \to 0$.

Using Theorems 3 and 7 we obtain necessary and sufficient conditions that $U \in \mathcal{D}'_{L^P}$ be the boundary value of a function $f(z)$ which is analytic in $G_{(1,\cdots,1)}$ and is bounded as in (4) for $z \in G_{(1,\cdots,1)}$ and $A = (0, \cdots, 0)$. We denote such functions by the symbol H^+.

Theorem 8. *$U \in \mathcal{D}'_{L^P}$, $1 \leq P \leq 2$, is the $\mathcal{S}'$ boundary value of a function $f(z) \in H^+$ if and only if*

$$\left\langle U, \prod_{j=1}^{n} \frac{1}{t_j - z_j} \right\rangle = 0, \; z \in G_\delta, \; \delta \neq (1, \cdots, 1).$$

The proofs of these results are similar in construction to those used for the one dimensional case. For $U \in \mathcal{D}'_{L^P}$, $1 \leq P \leq 2$, one can also define a generalized Poisson integral which is an n harmonic function of z and which converges in $\mathcal{S}'$ to the sum of two elements of $\mathcal{S}'$. For some further results concerning distributional boundary values in $\mathcal{D}'_{L^P}$ as a subspace of $\mathcal{S}'$ we refer to Carmichael [10].

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