189. A Note on a Paper of Farkas

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H. E. Rauch [1] and H. M. Farkas [2] discussed analytic submanifolds of Teichmüller space, and in relation to these studies Farkas [3] pointed out the following theorem:

Let S be a compact Riemann surface of genus $g \geqq 4$. Let q be a Weierstrass point on S whose Weierstrass sequence begins with 3. Then 4 is a gap at q.

In this paper we shall prove the following more general theorem.

Theorem. *Let S be a compact Riemann surface of genus $g > r(r-1)/2$, where $r (1 < r < g)$ is an integer. Let q be a Weierstrass point on S whose Weierstrass sequence begins with r. Then $r+1$ is a gap at q.*

First we recall some definitions and results from the theory of compact Riemann surfaces.

There are exactly g orders n_i, $0 < n_1 < n_2 < \cdots < n_g < 2g$, that can be specified at each point p on S such that no meromorphic function exists having as its only singularity a pole of order n_i at p. The sequence $(n_1, n_2, \cdots, n_g)$ is called then a gap sequence at p. Given a point p on S, its gap sequence is $(1, 2, \cdots, q)$ in general; however, there do exist points on S whose gap sequences omit some of these numbers. These points are called Weierstrass points. In other words, the gap sequence for a Weierstrass point omits an integer n, $2 \leqq n \leqq g$. The complement of the gap sequence in the sequence of integers $(1, 2, \cdots, 2g)$ is called the Weierstrass sequence.

Lemma. *If there is a Weierstrass point on S whose Weierstrass sequence contains $r, r+1, \cdots, r+m$, then*

$$(t+1)[(r-1)-tm/2] \geqq g$$

where t is the smallest integer which satisfies $t \geqq (r-1)/m$.

Proof. The integers $r, r+1, \cdots, r+m$ form the module whose elements are not gaps. Hence the gaps must be contained in the set of remaining natural numbers, which are $1, 2, \cdots, r-1$; $r+m+1$, $r+m+2, \cdots, 2r-1$; $\cdots$; $\cdots tr-1$; where t is the smallest integer which satisfies $t \geqq (r-1)/m$. While as is well known, the number of gaps is exactly g, so we have

$$(r-1)+(r-m-1)+\cdots+(r-tm-1) \geqq g$$

that is,

$$(t+1)[(r-1)-tm/2] \geqq g.$$

Proof of the theorem. Assume $r+1$ is not a gap. Putting $m=1$ in lemma, then we must have $r(r-1)/2 \geqq g$, which contradicts with our hypothesis.

References

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[3] ——: Weierstrass points and analytic submanifolds of Teichmüller space. Proc. Amer. Math. Soc., **20**, 35–38 (1969).

[4] G. Springer: Introduction to Riemann Surfaces. Addison-Wesley (1957).