184. *On the Topological Entropy of a Dynamical System*

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§ 1. **Preliminaries.** Let φ be a homeomorphism from a compact space X onto itself. If α is any open cover of X, we let $N(\alpha)$ be the number of members in a subcover of α of minimal cardinality. As in [1], the limit exists in the following definition:

$$h(\alpha, \varphi)=\lim_{n\to\infty}\frac{1}{n}\log N(\vee_{i=0}^{n-1}\varphi^i\alpha)^{*)}$$

and the topological entropy $h(\varphi)$ of φ is defined as $h(\varphi)=\sup h(\alpha, \varphi)$, where the supremum is taken over all open covers of X.

Let $\{\varphi_t\}$ be a homeomorphic flow on a compact space X. It was conjectured in [1] that

(F) $\qquad h(\varphi_t)=|t|\,h(\varphi_1)$ for all t.**)

In this paper, we will give a proof for (F) under the assumption that X is a compact metric space and that $\{\varphi_t\}$ is a dynamical system: namely,

(C) $\qquad \varphi_t x=\varphi(t, x)$ is continuous in the pair of variables t, x.

For later use, we will mention the well-known properties of the topological entropy [1]:

(1.1) $\qquad$ if $\alpha<\beta$ then $h(\varphi, \alpha)\leqslant h(\varphi, \beta)$

(1.2) $\qquad h(\varphi_k)=|k|\,h(\varphi_1)$ for integer k.

§ 2. **The theorem. Theorem.** If X is a compact mentric space and $\{\varphi_t\}$ is a dynamical system. Then

(F) $\qquad h(\varphi_t)=|t|\,h(\varphi_1)$ for any real t.

Proof. For any pair $\varepsilon_1>\varepsilon_2>0$, let α_i be the set of all open spheres of radius ε_i, $i=1, 2$. Then $\alpha_1<\alpha_2$.

Put $A_t=\{x\,|\,d(\varphi_s x, x)<\varepsilon_1-\varepsilon_2$ for $|s|\leqslant t\}$, where d is the distance of X. Then, by the continuity of $\{\varphi_t\}$, the set A_t is an open set, and $A_t\subset A_{t'}$ if $t>t'$, and moreover, $\bigcup_{t>0}A_t=X$. Thus, by the compactness of X, there exists a positive real number t_0 satisfying $A_{t_0}=X$.

If t, t' are arbitrary pair of positive numbers and T is an arbitrary large positive number, there exist positive integers p, n and m such that $t/p\leqslant t_0$,

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*) As in [1], we write $\alpha\vee\beta=\{U\cap V : U\in\alpha\ V\in\beta\}$ and we write $\alpha>\beta$ to mean that α is a refinement of β.

**) On the measure theoretic entropy (*F*) is proved in [2]; much simpler proof is given in [3].

$$T-t/p\leqslant(m-1)t/p<T \quad \text{and} \quad T-t'\leqslant(n-1)t'<T.$$

Consequently, there exists a subsequence $\{m_k\}_{k=0}^{n-1}$ of $1,2,\cdots,m$ such that

$$|kt'-m_k(t/p)|\leqslant t_0 \quad \text{for} \quad k=1,2,\cdots,n-1.$$

We will show that

$$N(\alpha_1\vee\varphi_{t'}\alpha_1\vee\cdots\vee\varphi_{(n-1)t'}\alpha_1)\leqslant N(\alpha_2\vee\varphi_{m_1(t/p)}\alpha_2\vee\cdots\vee\varphi_{m_{n-1}(t/p)}\alpha_2).$$

Let $\{A_1,A_2,\cdots,A_r\}$ be a minimal subover of

$$\alpha_2\vee\varphi_{m_1(t/p)}\alpha_2\vee\cdots\vee\varphi_{m_{n-1}(t/p)}\alpha_2.$$

A set A_i is written

$$A_i=A_0^{(i)}\cap\varphi_{m_1(t/p)}A_1^{(i)}\cap\cdots\cap\varphi_{m_{(n-1)}(t/p)}A_{n-1}^{(i)}$$

$$A_k^{(i)}\in\alpha_2 \quad i=1,2,\cdots,r. \quad k=0,1,2,\cdots,n-1.$$

Now for each $A_k^{(i)}$ there exists a $B_k^{(i)}$ which belongs to α_1 and has the same center as the sphere $A_k^{(i)}$.

It follows from $|m_k(t/p)-kt'|\leqslant t_0$ that

$$\varphi_{m_k(t/p)-kt'}A_k^{(i)}\subset B_k^{(i)} \quad \text{and} \quad \varphi_{m_k(t/p)}A_k^{(i)}\subset\varphi_{kt'}B_k^{(i)}.$$

Thus $B_0^{(i)}\cap\varphi_{t'}B_1^{(i)}\cap\cdots\cap\varphi_{(n-1)t'}B_{n-1}^{(i)}\supset A_0^{(i)}\cap\varphi_{m_1(t/p)}A_1^{(i)}\cap\cdots$

$$\cdots\cap\varphi_{m_{n-1}(t/p)}A_{n-1}^{(i)} \quad \text{for} \quad i=1,2,\cdots,r.$$

We thus obtain the relation

$$\text{(2.1)} \qquad \log N(\alpha_1\vee\varphi_{t'}\alpha_1\vee\cdots\vee\varphi_{(n-1)t'}\alpha) \leqslant\log N(\alpha_2\vee\varphi_{(t/p)}\alpha_2\vee\varphi_{2(t/p)}\alpha_2\vee\cdots\vee\varphi_{m-1(t/p)}\alpha_2).$$

Since $\lim_{T\to\infty}T/n=t'$ and $\lim_{T\to\infty}T/m=t/p$, by dividing both sides of (2.1) by T and then letting $T\to\infty$, we obtain the relation

$$\frac{1}{t'}h(\varphi_{t'},\alpha_1)\leqslant\frac{1}{(t/p)}h(\varphi_{t/p},\alpha_2).$$

Since $\varepsilon_1>\varepsilon_2>0$ are arbitrary, it follows from the definition of the topological entropy that

$$\frac{1}{t'}h(\varphi_{t'},\alpha_1)\leqslant\frac{1}{(t/p)}h(\varphi_{t/p})$$

$$\text{and} \quad \frac{1}{t'}h(\varphi_{t'})\leqslant\frac{1}{(t/p)}h(\varphi_{t/p})$$

By the formular (1.2), we then get $\frac{1}{t'}h(\varphi_{t'})\leqslant\frac{1}{t}h(\varphi_t)$. Reversing the role of t and t' in the argument above, we also obtain $\frac{1}{t'}h(\varphi_{t'})\geqslant\frac{1}{t}h(\varphi_t)$; therefore, it follows that $\frac{1}{t}h(\varphi_t)=\frac{1}{t'}h(\varphi_{t'})$ for any pair $t,t'>0$. In particular, $\frac{1}{t}h(\varphi_t)=h(\varphi_1)$ for any $t>0$.

Since in general $h(\varphi)=h(\varphi^{-1})$ holds for a homeomorphism we finally get $\frac{1}{|t|}h(\varphi_t)=h(\varphi_1)$ for all real $t(\neq 0)$. q.e.d.

Finally we give a natural definition for the entropy of a dynamical system.

Definition. The entropy of a dynamical system $\{\varphi_t\}$ is defined to be $h(\varphi_1)$.

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References

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[2] L. M. Abramov: On the entropy of a flow. Dok. Akad. Nauk. SSSR., **128**, 873–875 (1959).

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