11. On Generalized Integrals. VI

Restrictions of (E.R.) Integral. I

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As it is already known, the set of all $(E.R.)$ integrable functions is very large. For example, it is proved, in studies of the A-integral (which coincides with the special $(E.R.)$ integral), that every continuous function whose product with any A-integrable function is A-integrable, is constant [1], and that in the set of all those A-integrable functions f for which the indefinite integral $A(x)=(A)\int_a^x f(x)dx$ is defined,[1] $A(x)$ cannot be the indefinite integral of only one function, to within a set of measure zero, i.e. there is no one-to-one correspondence between a function and its indefinite A-integral [8]. For this reason, there arose the problem of specialization of the A-integral and the $(E.R.)$ integral (see [5], [2], [7], [10], [4], [9]). On the other hand, in connection with the Denjoy integral defined as an extension of the Lebesgue integral, we have seen that for a function $f(x)$ Denjoy-integrable in the general sense, there exists some φ for which $f(x)$ is $(E.R.\ \varphi)$ integrable and both integrals are given as limit of the same approximating sums (see [3], V[2], Theorem 10). We now define, in this paper, the integrals, called $(E.R.\ \varphi)_2$ (resp. $(E.R.\ \varphi)_3$) integral, which are considered as specializations of Denjoy integral-type in the general (resp. special) sense of the $(E.R.\ \varphi)$ integral, and prove that a function $f(x)$ Denjoy-integrable in the general (resp. special) sense is also $(E.R.\ \varphi)_2$ (resp. $(E.R.\ \varphi)_3$) integrable for $\varphi(=\varphi(f))$ reasonably chosen, and both integrals coincide (Theorem 11).

We conserve the terminologies and the notation of the preceding papers I–V [6].

9. Restrictions of (E.R.) integrals (1). Let $\varphi(x)$ be a positive, Lebesgue-integrable function in a finite or infinite interval $[a, b]$. Denote the set of all measurable functions in $[a, b]$, by $\mathcal{M}$, or, for the purpose of calling special attention to the interval $[a, b]$, by $\mathcal{M}\ (a, b)$. Before defining integrals of the new sense, we first consider the following conditions instead of $[\gamma(\varphi)]$, where $[\gamma(\varphi)]$ is one of the

1) In general, the existence of the A-integral of $f(x)$ on $[a, b]$ does not imply its existence on $[c, d]\subseteq[a, b]$.

2) The reference number indicates the number of the Note.

conditions introduced, in IV, to define the $(E.R.\ \varphi)$ integral, precisely, the system of neighbourhoods $\{V_\varphi(A, \varepsilon; f)\}$ in the ranked space $\{M, \varphi\}$:

$[\gamma_1(\varphi)]$ For every interval $[c, d] \subseteq [a, b]$, holds

$$\left|\int_c^d [r(x)]^{k\varphi(x)} dx\right| < \varepsilon \qquad \text{for each } k > 0.$$

$[\gamma_2(\varphi)]$ (resp. $[\gamma_2^*(\varphi)]$) For every sequence $\{[a_j, b_j]\}$ of non-overlapping intervals such that $a_j \in A$ and $b_j \in A$ (resp. at least one of a_j, b_j belongs to A) for each j, holds

$$\left|\sum_j \int_{a_j}^{b_j} [r(x)]^{k\varphi(x)} dx\right| < \varepsilon \qquad \text{for each } k > 0.$$

Definition 5. For $i=1, 2, 3$, the neighbourhood $V_\varphi^{(i)}(A, \varepsilon; f)$ of $f \in \mathcal{M}(a, b)$ in $\mathcal{M}(a, b)$, where A is a closed subset of $[a, b]$ with mes $A > 0$ and ε is a positive number, is the set of all $g \in \mathcal{M}(a, b)$ such that $g = f + r$, where r satisfies the following conditions respectively:

$[\alpha(\varphi)]$, $[\beta(\varphi)]$ and $[\gamma_1(\varphi)]$, when $i=1$,

$[\alpha(\varphi)]$, $[\beta(\varphi)]$, $[\gamma_1(\varphi)]$ and $[\gamma_2(\varphi)]$, when $i=2$,

$[\alpha(\varphi)]$, $[\beta(\varphi)]$ and $[\gamma_2^*(\varphi)]$, when $i=3$.

If there is no ambiguity about φ, we simply write $V^{(i)}(A, \varepsilon; f)$ for $V_\varphi^{(i)}(A, \varepsilon; f)$. We denote the space endowed with the neighbourhoods $V_\varphi^{(i)}(A, \varepsilon; f)$ by $\{\mathcal{M}, \varphi, i\}$ or $\{\mathcal{M}(a, b), \varphi, i\}$ and introduce the ranks on the spaces as in $\{\mathcal{M}(a, b), \varphi\}$ [IV]. Then, the spaces become ranked spaces. When $\varphi(x)=1$, we write $V^{(i)}(A, \varepsilon; f)$, $\{\mathcal{M}, i\}$ etc. for $V_\varphi^{(i)}(A, \varepsilon; f)$, $\{\mathcal{M}, \varphi, i\}$ etc. respectively.

In the ranked space $\{\mathcal{M}, \varphi, i\}$, we have first of all that:

Lemma 27. *For $i=1, 2, 3$, if $V_\varphi^{(i)}(A_n, \varepsilon_n; f)$ is a fundamental sequence, then $\{V_\varphi^{(i)}(A_n^*, \varepsilon_n; f)\}$, where $A_n^* = \bigcap_{m=n}^{\infty} A_m$, is also a fundamental sequence such that $V_\varphi^{(i)}(A_n^*, \varepsilon_n; f) \supseteq V_\varphi^{(i)}(A_n, \varepsilon_n; f)$ for each n.*

Lemma 28. *For $i=1, 2, 3$, if $\{\lim_n f_n\} \ni f$ in $\{\mathcal{M}(a, b), \varphi, i\}$, then $\{\lim_n f_n\} \ni f$ in $\{\mathcal{M}(c, d), \varphi, i\}$ for every $[c, d] \subseteq [a, b]$.*

Given two ranked spaces $\mathcal{R}_1$, $\mathcal{R}_2$ defined on the same set R, we say that $\mathcal{R}_1$ is *finer* than $\mathcal{R}_2$ if $\{\lim_n p_n\} \ni p$ in $\mathcal{R}_1$ implies $\{\lim_n p_n\} \ni p$ in $\mathcal{R}_2$.

Lemma 29. *For $i=1, 2, 3$, $\{\mathcal{M}, \varphi, i\}$ is finer than $\{\mathcal{M}, \varphi\}$.*

Proof. If $\{\lim_n f_n\} \ni f$ in $\{\mathcal{M}, \varphi, i\}$, then according to Lemma 27 there is a fundamental sequence $\{V_\varphi^{(i)}(A_n, \varepsilon_n; f)\}$ such that $V_\varphi^{(i)}(A_n, \varepsilon_n; f) \ni f_n$ and $A_n \subseteq A_{n+1}$. Since $A_n \subseteq A_{n+1}$ and $\varepsilon_n \downarrow 0$, $\{V_\varphi(A_n, \varepsilon_n; f)\}$ is also fundamental in $\{\mathcal{M}, \varphi\}$, and so, by reason of $V_\varphi(A_n, \varepsilon_n; f) \supseteq V_\varphi^{(i)}(A_n, \varepsilon_n; f)$, it follows that $\{\lim_n f_n\} \ni f$ in $\{\mathcal{M}, \varphi\}$.

Therefore, it follows from IV, Lemma 21 that

Lemma 30. *For $i=1, 2, 3$, if $\{f_n\}$ is an r-converging sequence in $\{\mathcal{M}, \varphi, i\}$, then the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists and $\{\lim_n f_n\}$ is the set consisting of f alone.*

Lemma 31. *For $i=1,2$, $\{\mathcal{M},\varphi,i+1\}$ is finer than $\{\mathcal{M},\varphi,i\}$.*

The proof is similar to that of Lemma 29.

In the same way as III, Lemma 13, IV, Lemma 23, we get the following two lemmas respectively:

Lemma 32. *For $i=1,2,3$, if $f\in\{\lim_n f_n\}$ and $g\in\{\lim_n g_n\}$ in $\{\mathcal{M},\varphi,i\}$, then we have $\alpha f+\beta g\in\{\lim_n(\alpha f_n+\beta g_n)\}$ in $\{\mathcal{M},\varphi,i\}$ for any pair, α and β, of real numbers.*

Lemma 33. *For $i=1,2,3$, $Cl_r(Cl_r\mathcal{S})=Cl_r\mathcal{S}$ holds in $\{M,\varphi,i\}$ for every subset $\mathcal{S}$ of $\mathcal{M}$.*

As in the preceding papers, let us denote by $\mathcal{E}$ or $\mathcal{E}(a,b)$ the set of all step functions on $[a,b]$. We consider the set of such functions which are defined as r-limits of the sequences $\{f_n\}$ of points of $\mathcal{E}$ in $\{\mathcal{M}(a,b),\varphi,i\}$, that is, $\mathrm{Cl}_r\mathcal{E}$ in $\{\mathcal{M}(a,b),\varphi,i\}$, and denote the set by $\boldsymbol{K}(\varphi,i)$ or $\boldsymbol{K}((a,b),\varphi,i)$. Then, a function $f\in\boldsymbol{K}((a,b),\varphi,i)$ is said to be $(E.R.\,\varphi)_i$ *integrable* in $[a,b]$. When $\varphi(x)=1$, we write $\boldsymbol{K}(i)$, $\boldsymbol{K}((a,b),i)$ for $\boldsymbol{K}(\varphi,i)$, $\boldsymbol{K}((a,b),\varphi,i)$ respectively, and call the $(E.R.\,\varphi)_i$ integrable function, the $(E.R.)_i$ integrable function.

We obtain, from Lemma 32 and Lemma 28, Proposition 19 and Proposition 20 respectively.

Proposition 19. *For $i=1,2,3$, $\boldsymbol{K}(\varphi,i)$ is a vector space.*

Proposition 20. *For $i=1,2,3$, if $f(x)$ is $(E.R.\,\varphi)_i$ integrable on $[a,b]$, $f(x)$ is also $(E.R.\,\varphi)_i$ integrable on $[c,d]$ for all $[c,d]\subseteq[a,b]$.*

Let us consider, as in IV, the mapping Tf of $\mathcal{M}(a,b)$ onto $\mathcal{M}(\alpha,\beta)$ defined in such a way that

$$Tf(y)=f(\Phi^{-1}(y))(\Phi^{-1}(y))' \qquad (y\in[\alpha,\beta]),$$

where $y=\Phi(x)$, $x\in[a,b]$, is the indefinite integral of $\varphi(x)$ such that $\Phi(a)=\alpha$ and $\Phi(b)=\beta$, and Φ^{-1} is the inverse of Φ. Then, we have:

Lemma 34. *For $i=1,2,3$, if $V_\varphi^{(i)}(A,\varepsilon;f)$ is a neighbourhood in $\{\mathcal{M}(a,b),\varphi,i\}$ then $T(V_\varphi^{(i)}(A,\varepsilon;f))$ is a neighbourhood in $\{\mathcal{M}(\alpha,\beta),i\}$, and*

$$T(V_\varphi^{(i)}(A,\varepsilon;f))=V^{(i)}(\Phi(A),\varepsilon;Tf).$$

The same is true of T^{-1}.

Lemma 35. *For $i=1,2,3$, $V_\varphi^{(i)}(A,\varepsilon;f)$ is a neighbourhood of f of rank n in $\{\mathcal{M}(a,b),\varphi,i\}$ if and only if $T(V_\varphi^{(i)}(A,\varepsilon;f))$ is a neighbourhood of Tf of rank n in $\{\mathcal{M}(\alpha,\beta),i\}$.*

Consequently, on account of IV, Lemma 16, we get the following proposition.

Proposition 21. *For $i=1,2,3$, the mapping T is an r-isomorphism of $\{\mathcal{M}(a,b),\varphi,i\}$ onto $\{\mathcal{M}(\alpha,\beta),i\}$.*

From this, it follows that:

Proposition 22. *For $i=1,2,3$, $T(\boldsymbol{K}((a,b),\varphi,i))=\boldsymbol{K}((\alpha,\beta),i)$.*

Proposition 23. $\mathcal{L}\subseteq\boldsymbol{K}(\varphi,3)\subseteq\boldsymbol{K}(\varphi,2)\subseteq\boldsymbol{K}(\varphi,1)\subseteq\boldsymbol{K}(\varphi)$.

Proof. It is easy, by Lemma 29 and Lemma 31, to see that $K(\varphi,3)\subseteq K(\varphi,2)\subseteq K(\varphi,1)\subseteq K(\varphi)$. $\mathcal{L}\subseteq K(\varphi,3)$ results from that $\lim_{k\to\infty}\int_a^b |[f(x)]^k - f(x)|\,dx=0$ and for $f\in\mathcal{L}$, there exists a sequence of step functions which converges in measure to f.

Proposition 24. *For $i=1,2,3$, $Cl_r(\mathcal{L})$ in $\{\mathcal{M},\varphi,i\}$ coincides with $K(\varphi,i)$.*

This results from Proposition 23 and Lemma 32. Furthermore, we have:

Proposition 25. *For $i=1,2,3$, if $f(x)$ is a $(E.R.\ \varphi)_i$ integrable function on $[a,c]$ and $[c,b]$, then $f(x)$ is also $(E.R.\ \varphi)_i$ integrable on $[a,b]$, and we have*

$$(E.R.\ \varphi)\int_a^b f(x)dx=(E.R.\ \varphi)\int_a^c f(x)dx+(E.R.\ \varphi)\int_c^b f(x)dx.$$

An important set of $(E.R.\ \varphi)_i$ $(i=2,3)$ integrable functions is shown in the following theorem, which is more precise than Theorem 10.

Theorem 11. *If $f(x)$ is a general (resp. special) Denjoy-integrable function in a finite interval $[a,b]$, then there is a positive Lebesgue-integrable function $\varphi(x)$ in $[a,b]$ for which $f(x)$ is $(E.R.\ \varphi)_2$ (resp. $(E.R.\ \varphi)_3$) integrable in $[a,b]$ and we have*

$$(D)\int_a^b f(x)dx\ \ (resp.\ (D_*)\int_a^b f(x)dx)=(E.R.\ \varphi)\int_a^b f(x)dx.$$

Proof. According to V, Theorem 9, for the function $f(x)$, there exists a monotone increasing sequence $\{F_n\}$, with union $[a,b]$, of closed sets such that: (i) $f(x)$ is Lebesgue-integrable on each F_n,

(ii) $\{F_n\}$ possess the properties $[C]$ and $[D]$ (resp. $[D_*]$)[3] for f,

(iii) $(D)\int_a^b f(x)dx=\lim_{n\to\infty}\int_{F_n} f(x)dx$ holds.

We define, for $\{F_n\}$, a monotone increasing sequence $\{A_n\}$ of closed sets so as to satisfy the following conditions: $A_n\subseteq F_n$, $A_{n+1}\setminus A_n$ is a closed set, $\lim_{n\to\infty}\text{mes}\,A_n=b-a$, $\int_{F_n\setminus A_n}|f(x)|\,dx<2^{-(n+1)}$ and $f(x)$ is bounded on every A_n. Let $\tau(x)$ be the mapping of $\cup A_n$ onto $[0,b-a)$ defined by the method of V, Lemma 26 for $\{A_n\}$. Then, $\tau(x)$ is a one-to-one mapping except for a set N of measure zero with mes $N^*=0$, where $N^*=\tau(N)$. Put $f^*(y)=f(\tau^{-1}(y))$ on $[0,b-a)\setminus N^*$ and zero elsewhere. Then, for $f^*(y)$, there exists, according to V, Lemma 24, a function $u(y)$ in $[0,b-a)$ with the properties 1) and 2) of V, Lemma 24. Put $\psi(y)=e^{u(y)-w(y)}$, where $w(y)=e^{u(y)}$. Moreover, put $f_I^*(y)=f(\tau^{-1}(y))$ on $\tau(\cup A_n\cap I)\setminus N^*$ and zero elsewhere, and put $\alpha_n=\text{mes}\,A_n$. Then, in a similar manner as in the proof of V, Proposition 17, we may choose a

3) For the definition, see V [6].

subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that:

(i) $u(\alpha_{n_i})e^{-u(\alpha_{n_i})}<2^{-(i+4)}$ and $\int_{\alpha_{n_i}}^{b-a}\psi(y)dy<2^{-(i+4)}$,

(ii) $\left|\int_{\alpha_{n_i}}^{y} f_I^*(y)\,dy\right|<2^{-(i+4)}$, $\alpha_{n_i}<y<b-a$,

holds for every interval I in $[a, b]$,

(iii) $\left|\sum_j \int_{\alpha_{n_i}}^{y} f_{I_j}^*(y)\,dy\right|<2^{-(i+4)}$, $\alpha_{n_i}<y<b-a$,

holds for every sequence $\{I_j\}$ of non-overlapping intervals such that for each j, the endpoints (resp. at least one of endpoints) belong to F_{n_i}. Hence, as in the proof of V, Lemma 25, if we put $g_{I,n_i}^*(y)=f(\tau^{-1}(y))$ on $\tau(A_{n_i}\cap I)\backslash N^*$ and zero elsewhere, where I is an interval in $[a, b]$, we see that:

$[\alpha(\psi)]$ $|g_{I,n_i}^*(y)-f_I^*(y)|=0$ for every $y\in[0,\alpha_{n_i}]$,

$[\beta(\psi)]$ $k\int_{\{y;\,|g_{I,n_i}^*(y)-f_I^*(y)|>k\psi(y)\}}\psi(y)dy<2^{-(i+2)}$ for each $k>0$,

$[\gamma_1(\psi)]$ $\left|\int_0^{b-a}[g_{I,n_i}^*(y)-f_I^*(y)]^{k\psi(y)}dy\right|<2^{-(i+2)}$ for each $k>0$,

$[\gamma_2(\psi)]$ (resp. $[\gamma_2^*(\psi)]$) for every sequence $\{I_j\}$ of non-overlapping intervals such that for each j, the endpoints (resp. at least one of endpoints) belong to F_{n_i}, we have

$$\left|\sum_j \int_0^{b-a}[g_{I_j,n_i}^*(y)-f_{I_j}^*(y)]^{k\psi(y)}dy\right|<2^{-(i+2)}$$

for each $k>0$.

From this, it follows, in the same way as in the proof of V, Proposition 17, that for a positive Lebesgue-integrable function $\varphi(x)=\psi(\tau(x))$ defined in $[a, b]$, $\{V_\varphi^{(2)}(F_{n_i}, \varepsilon_i; f)\}$ (resp. $\{V_\varphi^{(3)}(F_{n_i}, \varepsilon_i; f)\}$), $\varepsilon_i=2^{-i}$, is a fundamental sequence in $\{\mathcal{M}(a, b), \varphi, 2\}$ (resp. $\{\mathcal{M}(a, b), \varphi, 3\}$) such that $V_\varphi^{(2)}(F_{n_i}, \varepsilon_i; f)$ (resp. $V_\varphi^{(3)}(F_{n_i}, \varepsilon_i; f)) \ni f_{n_i}$ for each i, where f_{n_i} is a function defined as follows: $f_{n_i}(x)=f(x)$ on F_{n_i} and zero elsewhere. This indicates that $\{\lim_i f_{n_i}\}\ni f$ in $\{\mathcal{M}(a, b), \varphi, 2\}$ (resp. $\{\mathcal{M}(a, b), \varphi, 3\}$). Thus, in virtue of Proposition 23, our assertion follows.

Remark 2. For the fundamental sequences $\{V_\varphi^{(2)}(F_{n_i}, \varepsilon_i; f)\}$ and $\{V_\varphi^{(3)}(F_{n_i}, \varepsilon_i; f)\}$ defined in the proof of Theorem 11, holds $\bigcup_i F_{n_i}=[a, b]$.

References

[1] I. L. Bondi: On the property of A-integrable function. Uspehi Mat. Nauk, **18** (109), 145–150 (1963).

[2] ——: A-integrability in the narrow sense. Mat. Sb., **61** (103), 121–146 (1963).

[3] K. Fujita: On indefinite ($E. R.$)-integrals. I, II. Proc. Japan Acad., **41**, 686–695 (1965).

[4] ——: On the so-called fundamental theorem of integration. Proc. Japan Acad., **42**, 339–343 (1966).

[5] S. Nakanishi: Sur la dérivation de l'intégrale (*E.R.*) indéfinie. I. Proc. Japan Acad., **34**, 199-204 (1958).
[6] ——: On generalized integrals. I-V. Proc. Japan Acad., **44**, 133-138, 225-230, 904-909 (1968), **45**, 86-91, 374-379 (1969).
[7] H. Okano: Sur une généralisation de l'intégrale (*E.R.*) et un théorème général de l'intégration par parties. Jour. Math. Soc. Japan, **14**, 430-442 (1962).
[8] I. A. Vinogradova: On the representation of a measurable function by an indefinite *A*-integral. Izv. Akad. Nauk SSSR Ser. Mat., **26**, 581-604 (1962).
[9] ——: A restriction of the *A*-integral. Math. Sb., **72** (114), 365-387 (1967).
[10] F. S. Vaher: The general form of a linear functional on the Banach space of analytic functions and the *A*-integral. Dokl. Akad. Nauk SSSR, **166**, 518-521 (1966).