4. *On wM-Spaces. II*

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1. Introduction. This is the continuation of our previous paper [6]. The purpose of this paper is to study metrizability of wM-spaces and to give a solution to a problem under what conditions a wM-space is an M-space.

Definition. A topological space X has a $\bar{G}_\delta(k)$-diagonal ($G_\delta(k)$-diagonal, $k=1,2,\cdots$, if there exists a sequence $\{\mathfrak{V}_n\}$ of open coverings of X such that for distinct points x, y there exists some $\mathfrak{V}_m$ such that $y \notin \overline{\mathrm{St}^k(x, \mathfrak{V}_m)}$ ($y \notin \mathrm{St}^k(x, \mathfrak{V}_m)$).

By J. G. Ceder [5], a space X has a $G_\delta(1)$-diagonal ($=G_\delta$-diagonal in [4]) if and only if the diagonal Δ of $X\times X$ is a G_δ-subset of $X\times X$.

2. Metrizability of wM-spaces.

We shall prove some metrization theorems for wM-spaces.

Theorem 2.1. *In order that a space X be metrizable it is necessary and sufficient that X be a normal wM-space which has a $\bar{G}_\delta(1)$-diagonal.*

Proof. The necessity of the condition is obvious. To prove the sufficiency of the condition, let X be a normal wM-space with a decreasing sequence $\{\mathfrak{U}_n\}$ of open coverings of X satisfying (M_2), and suppose that X has a $\bar{G}_\delta(1)$-diagonal, that is, there exists a decreasing sequence $\{\mathfrak{V}_n\}$ of open coverings of X such that for distinct points x, y there exists some $\mathfrak{V}_n$ such that $y \notin \overline{\mathrm{St}(x, \mathfrak{V}_n)}$. Then clearly X is Hausdorff. Let us put $\mathfrak{W}_n = \mathfrak{U}_n \cap \mathfrak{V}_n, n=1,2,\cdots$. Then it is proved that $\{\mathrm{St}(x, \mathfrak{W}_n) \mid n=1,2,\cdots\}$ is a basis for neighborhoods at each point x of X. Indeed, if not, then there exist a point x_0 of X and an open subset U of X such that $x_0 \in U$ and $\mathrm{St}(x_0, \mathfrak{W}_n) - U \neq \emptyset$ for each n. Let $x_n \in \mathrm{St}(x_0, \mathfrak{W}_n) - U, n=1,2,\cdots$. Then by ($M_2$) the sequence $\{x_n\}$ has an accumulation point y which is contained in $X-U$. Since $x_0 \neq y$, we have $y \notin \overline{\mathrm{St}(x_0, \mathfrak{W}_k)}$ for some k, while $y \in \cap \overline{\mathrm{St}(x_0, \mathfrak{W}_n)}$. This is a contradiction, and hence $\{\mathrm{St}(x, \mathfrak{W}_n) \mid n=1,2,\cdots\}$ is a basis for neighborhoods at each point x of X. On the other hand, as is proved in our previous paper [6], every normal wW-space X is collectionwise normal (cf. [6, Theorem 2.4]). Hence, by a theorem of R. H. Bing [2], X is metrizable. Thus we complete the proof.

Theorem 2.2. *In order that a space X be metrizable it is neces-*

sary and sufficient that X be a wM-space which has a $\bar{G}_\delta(2)$-diagonal.

This theorem could be deduced from the following metrization theorem.

Theorem 2.3. *In order that a T_0 space X be metrizable it is necessary and sufficient that there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X such that $\{\mathrm{St}^2(x, \mathfrak{A}_n) \mid n=1,2\cdots\}$ is a basis for neighborhoods at each point x of X.*

Theorem 2.3 is essentially due to K. Morita [8, Theorem 4], and afterwards it is also proved by A. H. Stone [12, Theorem 1] and A. Arhangel'skii [1, Theorem 2]. But we shall give our proof for this theorem based on [6, Theorem 2.4].

Proof of Theorem 2.3. Since the condition is trivially necessary, we shall prove only the sufficiency of the condition. First we note that X is Hausdorff. Indeed, for distinct points x, y, one of them, say x, has a neighborhood $\mathrm{St}^2(x, \mathfrak{A}_n)$ not containing y, which implies $\mathrm{St}(x, \mathfrak{A}_n) \cap \mathrm{St}(y, \mathfrak{A}_n) = \emptyset$. Hence X is Hausdorff. We next show that X is normal. Let A and B be closed subsets of X such that $A \cap B = \emptyset$, and put

$$G_n = \bigcup\{\mathrm{St}(x, \mathfrak{A}_n) \mid x \in A,\ \mathrm{St}^2(x, \mathfrak{A}_n) \cap B = \emptyset\},$$
$$H_n = \bigcup\{\mathrm{St}(y, \mathfrak{A}_n) \mid y \in B,\ \mathrm{St}^2(y, \mathfrak{A}_n) \cap A = \emptyset\}$$

for each n. Then $A \subset \bigcup G_n$, $B \subset \bigcup H_n$ and $G_n \cap H_n = \emptyset$, $n=1, 2, \cdots$. Since we may assume that $\{\mathfrak{A}_n\}$ is decreasing, we have also $G_n \cap H_m = \emptyset$ for every m and n. Hence, if we put $P = \bigcup G_n$ and $Q = \bigcup H_n$, then P and Q are open subsets of X such that $A \subset P$, $B \subset Q$ and $P \cap Q = \phi$, which shows that X is normal. On the other hand, X is clearly a wM-space. Therefore by [6, Theorem 2.4] X is collectionwise normal. Consequently X is metrizable by a theorem of R. H. Bing [2]. Thus we complete the proof.

Proof of Theorem 2.2. The necessity of the condition is obvious. To prove the sufficiency of the condition, let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2), and suppose that X has a $\bar{G}_\delta(2)$-diagonal, that is, there exists a decreasing sequence $\{\mathfrak{B}_n\}$ of open coverings of X such that for distinct points x, y there exists some $\mathfrak{B}_n$ such that $y \notin \overline{\mathrm{St}^2(x, \mathfrak{B}_n)}$. Then clearly X is Hausdorff. Let us put $\mathfrak{W}_n = \mathfrak{A}_n \cap \mathfrak{B}_n$, $n=1, 2, \cdots$. Then, by the similar way as in the proof of Theorem 2.1, it is proved that $\{\mathrm{St}^2(x, \mathfrak{W}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods at each point x of X. Hence, by Theorem 2.3, X is metrizable. Thus we complete the proof.

From Theorem 2.1 (or 2.2), we can easily deduce a metrization theorem of A. Okuyama [10] and C. Borges [3]. In Theorems 2.1 and 2.2, we don't know whether a $\bar{G}_\delta(1)$-diagonal and a $\bar{G}_\delta(2)$-diagonal are replaced by a $G_\delta(1)$-diagonal and a $G_\delta(2)$-diagonal, respectively.

The following theorem is a consequence of a theorem of A. Okuyama [11, Theorem 3.6].

Theorem 2.4. *In order that a space X be a metrizable it is necessary and sufficient that X be a normal Hausdorff wM-space with a σ-locally finite net.*[1)]

Proof. The necessity of the condition is obvious. To prove the sufficiency of the condition, let X be a normal Hausdorff wM-space with a σ-locally finite net. Then by [6, Theorem 2.4] X is collectionwise normal. Further, as is shown by A. Okuyama [11], every collectionwise normal Hausdorff space with a σ-locally finite net is paracompact. Since every paracompact Hausdorff wM-space is an M-space, the theorem immediately follows from a theorem of A. Okuyama [11, Theorem 3.6]. Thus we complete the proof.

Finally, we shall state a metrization theorem based on symmetric neighborhoods.

Theorem 2.5. *In order that a T_0 space X be metrizable it is necessary and sufficient that each point x of X has a sequence $\{U_n(x) \mid n=1, 2, \cdots\}$ of symmetric neighborhoods such that $\{U_n^2(x) \mid n=1, 2, \cdots\}$ is a basis of neighborhoods at x.*

This theorem is easily proved by a theorem of J. Nagata [9, Theorem 1], but is also proved by Theorem 2.3 as follows:

Proof of Theorem 2.5. The necessity of the condition is obvious. To prove the sufficiency of the condition, suppose that each point x of a T_0 space X has a sequence $\{U_n(x)\}$ of symmetric neighborhoods such that $\{U_n^2(x)\}$ is a basis for neighborhoods at x, where we may assume that $\{U_n(x)\}$ is decreasing at each point x. Then it is proved that $\{U_n^4(x)\}$ is a basis for neighborhoods at each point x of X. Indeed, for given n and x, we can take p, q and r such that $p>q>r>n$, $U_r^2(x)\subset U_n(x)$, $U_q^2(x)\subset U_r(x)$, and $U_p^2(x)\subset U_q(x)$. Then clearly $U_p^4(x) \subset U_n(x)$, and hence $\{U_n^4(x)\}$ is a basis for neighborhoods at each point x. Now let us put $\mathfrak{A}_n=\{\text{Int } U_n(x) \mid x \in X\}$, $n=1, 2, \cdots$. Then $\text{St}^2(x, \mathfrak{A}_n) \subset U_n^4(x)$ for each n and x. Consequently by Theorem 2.3 X is metrizable.

Remark. K. Morita pointed out in Zbl., **78**, p. 361 (1958) that Nagata's theorem [9, Theorem 1] is easily proved by his metrization theorem [8, Theorem 4].

3. wM-spaces and M-spaces.

A wM-space X is not an M-space in general. Hence it is significant to study a problem under what conditions a wM-space X is an

1) The notion of net was introduced by A. Arhangel'skii in "An addition theorem for the weight of spaces lying compacta, Dokl. Akad. Nauk SSSR, **126**, 239-241 (1959)".

M-space. If X is an M-space, then there exists a normal sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_1), and hence the followings are valid.

(1) $\{\mathfrak{A}_n\}$ satisfies (M_2).

(2) $\bigcap \overline{\mathrm{St}^2(x, \mathfrak{A}_n)} = \bigcap \mathrm{St}(x, \mathfrak{A}_n)$ for each point x of X.

Conversely, we can prove the following

Theorem 3.1. *Let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying* (M_2). *If $\bigcap \overline{\mathrm{St}^2(x, \mathfrak{A}_n)} = \bigcap \mathrm{St}(x, \mathfrak{A}_n)$ for each point x of X, then X is an M-space.*

We shall prove Theorem 3.1 by the similar way as in the proof of [7, Theorem 6.1]. Before proving the theorem, we mention a lemma.

Lemma 3.2. *Let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying* (M_2). *If $\bigcap \overline{\mathrm{St}^2(x, \mathfrak{A}_n)} = \bigcap \mathrm{St}(x, \mathfrak{A}_n)$ for each point x of X, then for each k $\{\mathrm{St}^k(x, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods of $\bigcap \mathrm{St}(x, \mathfrak{A}_n)$.*

Proof. We prove the lemma by induction for k. For simplicity, we put $C(x) = \bigcap \mathrm{St}(x, \mathfrak{A}_n)$. Now suppose that $\bigcap \overline{\mathrm{St}^2(x, \mathfrak{A}_n)} = C(x)$. Then it is easily proved that $\{\mathrm{St}^2(x, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods of $C(x)$. Next, suppose that $\{\mathrm{St}^k(x, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods of $C(x)$ for some $k>2$. Then for any open subset U of X such that $C(x) \subset U$ there exist some m, n such that $m>n$, $\mathrm{St}^2(x, \mathfrak{A}_n) \subset U$ and $\mathrm{St}^k(x, \mathfrak{A}_m) \subset \mathrm{St}(x, \mathfrak{A}_n)$. Hence it follows that $\mathrm{St}^{k+1}(x, \mathfrak{A}_m) \subset U$. Thus we complete the proof.

Proof of Theorem 3.1. Suppose that $\bigcap \overline{\mathrm{St}^2(x, \mathfrak{A}_n)} = C(x)$ where $C(x) = \bigcap \mathrm{St}(x, \mathfrak{A}_n)$. Then by Lemma 3.2 $\{\mathrm{St}^2(x, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods of $C(x)$, and hence for given n and x there exists some m such that $\mathrm{St}^2(x, \mathfrak{A}_m) \subset \mathrm{St}(x, \mathfrak{A}_n)$. This shows that we can take $\{\mathrm{St}(x, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ as a basis for neighborhoods at each point x of X. We denote by $(X, \mathfrak{A})$ the space X with this new topology. For any subset A of X, let us put

$$\mathrm{Int}(A ; \mathfrak{A}) = \{x \mid \mathrm{St}(x, \mathfrak{A}_n) \subset A \text{ for some } n\}.$$

Then $\mathrm{Int}(A ; \mathfrak{A})$ is open in $(X, \mathfrak{A})$. Now we shall define that two points x and y are equivalent, i.e., $x \sim y$, if $y \in C(x)$. Then it is obvious that $x \sim x$ and that $x \sim y$ implies $y \sim x$. To prove transitivity of this relation, let $x \sim y$ and $y \sim z$. Then from $y \in C(x)$ and $z \in C(y)$ it follows that $z \in \mathrm{St}^2(x, \mathfrak{A}_n)$ for every n, and hence we obtain $z \in C(x)$, i.e., $x \sim z$. Let $X/\mathfrak{A}$ be a quotient space obtained from $(X, \mathfrak{A})$ by this equivalent relation, and let φ be a quotient map of $(X, \mathfrak{A})$ onto $X/\mathfrak{A}$. Then we have

$$\varphi^{-1}(\varphi(\mathrm{Int}(A ; \mathfrak{A}))) = \mathrm{Int}(A ; \mathfrak{A}).$$

Hence φ is an open continuous map of $(X, \mathfrak{A})$ onto $X/\mathfrak{A}$. We denote by ψ an identity map of X onto $(X, \mathfrak{A})$. Then ψ is continuous. Let us put $f=\varphi\circ\psi$, and $T=X/\mathfrak{A}$. Then we can prove that T is metrizable and $f: X\to T$ is closed. Indeed, let us put

$$\mathfrak{V}_n=\{\varphi(\mathrm{Int}(\mathrm{St}(x, \mathfrak{A}_n); \mathfrak{A}))\,|\, x\in X\},\quad n=1, 2, \cdots.$$

Then clearly $\mathfrak{V}_n, n=1, 2, \cdots$, are open coverings of T. Further, $\{\mathrm{St}^2(t, \mathfrak{V}_n)\,|\,n=1, 2, \cdots\}$ is a basis for neighborhoods at each point t of T. To show this, let V be any open subset of T containing a point t, and let $x_0\in\varphi^{-1}(t)$. Then $C(x_0)=\varphi^{-1}(t)\subset\varphi^{-1}(V)$, and hence by Lemma 3.2 there exists some $\mathfrak{A}_n$ such that $\mathrm{St}^4(x_0, \mathfrak{A}_n)\subset\varphi^{-1}(V)$. Since $\varphi^{-1}(\mathrm{St}^2(t, \mathfrak{V}_n))\subset\mathrm{St}^4(x_0, \mathfrak{A}_n)$, we obtain $\mathrm{St}^2(t, \mathfrak{V}_n)\subset V$, which shows that $\mathrm{St}^2(t, \mathfrak{V}_n)\,|\,n=1, 2, \cdots\}$ is a basis for neighborhoods at t. Consequently, by Theorem 2.3, T is metrizable. To prove the closedness of f, let A be any closed subset of X, and $t_0\in\overline{f(A)}$. Let $x_0\in f^{-1}(t_0)$. Since $\varphi(\mathrm{Int}(\mathrm{St}(x_0, \mathfrak{A}_n); \mathfrak{A})), n=1, 2, \cdots$, are open subsets of T containing t, we have $f(A)\cap\varphi(\mathrm{Int}(\mathrm{St}(x_0, \mathfrak{A}_n); \mathfrak{A}))\neq\emptyset$ for every n, which shows that $A\cap\mathrm{St}(x_0, \mathfrak{A}_n)\neq\emptyset$ for every n. Let $x_n\in A\cap\mathrm{St}(x_0, \mathfrak{A}_n)$. Then the sequence $\{x_n\}$ has an accumulation point y which is contained in $A\cap C(x_0)$. Hence we have $t_0=f(x_0)=f(y)\in f(A)$. This shows that f is closed. Finally it is obvious that $f^{-1}(t)$ is countably compact for each point t of T. Therefore, by a theorem of K. Morita [7], X is an M-space.

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