3. *On wM-Spaces. I*

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1. Introduction. The purpose of the present paper is to introduce the notion of wM-spaces, which is a generalization of M-spaces introduced by K. Morita [6], and to show some preperties of these spaces. For a sequence $\{\mathfrak{A}_n\}$ of open (or closed) coverings of a topological space X, we shall consider the following two conditions:

(M_1) If $\{K_n\}$ is a decreasing sequence of non-empty subsets of X such that $K_n \subset \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n and for some point x_0 of X, then $\cap \bar{K}_n \neq \emptyset$.

(M_2) If $\{K_n\}$ is a decreasing sequence of non-empty subsets of X such that $K_n \subset \mathrm{St}^2(x_0, \mathfrak{A}_n)$ for each n and for some point x_0 of X, then $\cap \bar{K} \neq \emptyset$.[1)]

A space X is an M-space if there exists a normal sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_1). A space X is an M^*-space ($M^\sharp$-space) if there exists a sequence $\{\mathfrak{F}_n\}$ of locally finite (closure preserving) closed coverings of X satisfying (M_1) (T. Ishii [2], F. Siwiec and J. Nagata [8]). A space X is a $w\Delta$-space if there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_1) (C. Borges [1]). As is shown by K. Morita [7], there exists an M^*-space which is locally compact Hausdorff but not an M-space. Further, in our previous paper [3], we proved that a normal space X is an M-space if and only if it is an M^*-space.

Now we shall define wM-spaces including all M-spaces, M^*-spaces and $M^\sharp$-spaces.

Definition. A space X is a wM-space if there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2).

In the above definition, we may assume without loss of generality that $\mathfrak{A}_{n+1}$ refines $\mathfrak{A}_n$ for each n.

As a remarkable property of a wM-space, we can prove that every normal wM-space is strongly normal, that is, collectionwise normal and countably paracompact (Theorem 2.4). This result plays an important role in metrizability of wM-spaces in the next paper. Throughout this paper we assume at least T_1 for every topological spaces unless otherwise specified.

1) For each positive integer k, $\mathrm{St}^k(x_0, \mathfrak{A}_n)$ denotes the iterated star of a point x_0 in each covering $\mathfrak{A}_n$.

We express our hearty thanks to Prof. K. Morita for his kind advices.

2. Some properties of wM-spaces.

Theorem 2.1. *For a space X, the following conditions are equivalent.*

(1) *X is a wM-space with a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying* (M_2).

(2) *There exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X such that, for any locally finite sequence $\{A_n\}$ of subsets of X, $\{\mathrm{St}(A_n, \mathfrak{A}_n) \mid n=1,2,\cdots\}$ is locally finite in X.*

(3) *There exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X such that, for any discrete sequence $\{x_n\}$ of points of X, $\{\mathrm{St}(x_n, \mathfrak{A}_n) \mid n=1,2,\cdots\}$ is locally finite in X.*

Proof. (1)→(2). Let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2). Then we can prove that, for any locally finite sequence $\{A_n\}$ of subsets of X, $\{\mathrm{St}(A_n, \mathfrak{A}_n)\}$ is locally finite in X. Indeed, if not, then for some locally finite sequence $\{A_n\}$ of subsets of X, $\{\mathrm{St}(A_n, \mathfrak{A}_n)\}$ is not locally finite in X. Hence there exists a point x_0 such that any neighborhood of x_0 intersects infinitely many elements of $\{\mathrm{St}(A_n, \mathfrak{A}_n)\}$. Therefore, for each n, we can select some positive integer $i(n)$ such that $\mathrm{St}(x_0, \mathfrak{A}_n) \cap \mathrm{St}(A_{i(n)}, \mathfrak{A}_{i(n)}) \neq \emptyset$, $n < i(n)$. Let $y_{i(n)} \in \mathrm{St}(x_0, \mathfrak{A}_n) \cap \mathrm{St}(A_{i(n)}, \mathfrak{A}_{i(n)})$. Then the sequence $\{y_{i(n)}\}$ has an accumulation point y_0 in X, and hence we can select a subsequence $\{y_{j(n)}\}$ of $\{y_{i(n)}\}$ such that $y_{j(n)} \in \mathrm{St}(y_0, \mathfrak{A}_n)$, $i(n)<j(n)$. Since $y_{j(n)} \in \mathrm{St}(A_{j(n)}, \mathfrak{A}_{j(n)}) \subset \mathrm{St}(A_{j(n)}, \mathfrak{A}_n)$, we have $A_{j(n)} \cap \mathrm{St}^2(y_0, \mathfrak{A}_n) \neq \emptyset$. Let $x_{j(n)} \in A_{j(n)} \cap \mathrm{St}^2(y_0, \mathfrak{A}_n)$. Then the sequence $\{x_{j(n)}\}$ has an accumulation point in X by (M_2), while it has no accumulation point in X by local finiteness of $\{A_{j(n)}\}$. This is a contradiction. Hence (2) holds.

(2)→(3). This implication is obvious.

(3)→(1). Let $\{\mathfrak{A}_n\}$ be a sequence of open coverings of X such that, for any discrete sequence $\{x_n\}$ of points of X, $\{\mathrm{St}(x_n, \mathfrak{A}_n)\}$ is locally finite in X. First, we prove that $\{\mathfrak{A}_n\}$ satisfies (M_1). To prove this, assume to be contrary. Then there exists a discrete sequence $\{x_n\}$ of points of X such that $x_n \in \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n and for some point x_0 of X. Since $x_0 \in \mathrm{St}(x_n, \mathfrak{A}_n)$ for each n, $\{\mathrm{St}(x_n, \mathfrak{A}_n)\}$ is not locally finite in X, while it is locally finite in X by our assumption. This is a contradiction. Hence $\{\mathfrak{A}_n\}$ satisfies (M_1). Next, we prove that $\{\mathfrak{A}_n\}$ satisfies (M_2). To prove this, assume to be contrary. Then there exists a discrete sequence $\{x_n\}$ of points of X such that $x_n \in \mathrm{St}^2(x_0, \mathfrak{A}_n)$ for each n and for some point x_0 of X. Since $\mathrm{St}(x_n, \mathfrak{A}_n) \cap \mathrm{St}(x_0, \mathfrak{A}_n) \neq \emptyset$, we can select a point $y_n \in \mathrm{St}(x_n, \mathfrak{A}_n) \cap \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n. Then the sequence

$\{y_n\}$ has an accumulation point in X by (M_1), while it has no accumulation point in X, because $\{\mathrm{St}(x_n, \mathfrak{A}_n)\}$ is locally finite in X. This is a contradiction. Hence (1) holds. Thus we complete the proof.

As the other characterizations of wM-spaces, we can prove the following

Theorem 2.2. *For a space X, the following conditions are equivalent.*

(1) *X is a wM-space.*

(2) *Each point x of X has a sequence $\{U_n(x)\}$ of symmetric neighborhoods (i.e., $y \in U_n(x)$ implies $x \in U_n(y)$) satisfying the condition $(*)$ below:*

$(*)$ *If $\{x_n\}$ is a sequence of points of X such that $x_n \in U_n^2(x_0)$ for each n and for some point x_0 of X, then the sequence $\{x_n\}$ has an accumulation point in X, where $U_n^2(x_0) = \cup\{U_n(y) \mid y \in U_n(x_0)\}$.*

(3) *Each point x of X has a sequence $\{U_n(x)\}$ of symmetric neighborhoods such that, for any locally finite sequence $\{A_n\}$ of subsets of X, $\{U_n(A_n) \mid n = 1, 2, \cdots\}$ is locally finite in X, where $U_n(A_n) = \cup\{U_n(y) \mid y \in A_n\}$.*

(4) *Each point x of X has a sequence $\{U_n(x)\}$ of symmetric neighborhoods such that, for any discrete sequence $\{x_n\}$ of points of X, $\{U_n(x_n) \mid n=1, 2, \cdots\}$ is locally finite in X.*

Proof. (1)→(2). Let X be a wM-space with a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2), and put $U_n(x) = \mathrm{St}(x, \mathfrak{A}_n)$ for each point x of X and for each n. Then $\{U_n(x) \mid n=1, 2, \cdots\}$ is a sequence of symmetric neighborhoods of x and satisfies $(*)$, because $U_n^2(x) = \mathrm{St}^2(x, \mathfrak{A}_n)$.

(2)→(3). This implication can be proved by the similar way as in the proof of the implication (1)→(2) in Theorem 2.1.

(3)→(4). This implication is obvious.

(4)→(1). Suppose that each point x of X has a sequence $\{U_n(x)\}$ of symmetric neighborhoods such that, for any discrete sequence $\{x_n\}$ of points of X, $\{U_n(x_n)\}$ is locally finite in X. Then it is easily verified that any sequence $\{x_n\}$ of points of X such that $x_n \in U_n(x_0)$ for some point x_0 of X and for each n has an accumulation point in X. Further, it is proved by induction for k that any sequence $\{x_n\}$ of points of X such that $x_n \in U_n^k(x_0)$ for some point x_0 of X and for each n has an accumulation point in X.[2)] Now let us put $\mathfrak{A}_n = \{\mathrm{Int}\, U_n(x) \mid x \in X\}$ for $n = 1, 2, \cdots$. Then $\{\mathfrak{A}_n\}$ satisfies (M_2), because $\mathrm{St}^2(x, \mathfrak{A}_n) \subset U_n^4(x)$. Hence (1) holds. Thus we complete the proof.

Theorem 2.3. *Every $M^\sharp$-space is a wM-space.*

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2) For a point x_0 of X and for each n, the sets $U_n^k(x_0)$, $k=2, 3, \cdots$, are defined inductively, i.e., $U_n^k(x_0) = \cup\{U_n(y) | y \in U_n^{k-1}(x_0)\}$.

Proof. Let X be an $M^{\sharp}$-space with a sequence $\{\mathfrak{F}_n\}$ of closure preserving closed coverings of X satisfying (M_1), where we may assume without loss of generality that $\{\mathfrak{F}_n\}$ is decreasing. Then for each $k \geqq 2$ it is easily proved that, if $\{K_n\}$ is a decreasing sequence of non-empty subsets of X such that $K_n \subset \mathrm{St}^k(x_0, \mathfrak{F}_n)$ for each n and for some point x_0 of X, then $\cap \bar{K}_n \neq \emptyset$. Let us put $\mathfrak{A}_n = \{\mathrm{Int}(\mathrm{St}(x, \mathfrak{F}_n)) \mid x \in X\}$ for each n. Then $\{\mathfrak{A}_n\}$ is a sequence of open coverings of X and satisfies (M_2), because $\mathrm{St}^2(x, \mathfrak{A}_n) \subset \mathrm{St}^4(x, \mathfrak{F}_n)$. Hence X is a wM-space. Thus we complete the proof.

In view of Theorem 2.3, all M- and M^*-spaces are also wM-spaces.

Now we shall show by an example that a $w\Delta$-space is not a wM-space in general, that is, the condition (M_1) does not imply the condition (M_2).

Example. (*A* $w\Delta$-space which is not a wM-space). Let R be the set of ordinals not greater than the first infinite ordinal ω, and let S be the set of ordinals not greater than the first uncountable ordinal Ω, each with the order topology. If we put $X = R \times S - \{(\omega, \Omega)\}$, then the space X is a locally compact Hausdorff $w\Delta$-space but is not a wM-space. Indeed, if we put

$$\mathfrak{A}_n = \{\{i\} \times S, \bigcup_{n \leqq j \leqq \omega} (\{j\} \times (S - \{\Omega\})) \mid 1 \leqq i < \omega\}$$

for each n, then $\{\mathfrak{A}_n\}$ satisfies (M_1). But, if we put $x_n = (n, \Omega)$, $n = 1, 2, \cdots$, then there is no sequence $\{\mathfrak{B}_n\}$ of open coverings of X such that $\{\mathrm{St}(x_n, \mathfrak{B}_n)\}$ is locally finite in X, and hence X is not a wM-space. Finally, it is obvious that X is a locally compact Hausdorff space.

Theorem 2.4. *Every normal wM-space is strongly normal, that is, collectionwise normal and countably paracompact.*

To prove Theorem 2.4, we use the following lemmas.

Lemma 2.5. *Let X be a wM-space with a sequence* $\{\mathfrak{A}_n\}$ *of open coverings of X satisfying* (M_2), *and let k be a positive integer such that* $k \geqq 3$. *If* $\{x_n\}$ *is a sequence of points of X such that* $x_n \in \mathrm{St}^k(x_0, \mathfrak{A}_n)$ *for each n and for some point* x_0 *of X, then the sequence* $\{x_n\}$ *has an accumulation point in X.*

This lemma immediately follows from (3) in Theorem 2.1 by induction for k.

Lemma 2.6. *Every wM-space is countably paracompact.*

Proof. Let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2), and let $\{G_n\}$ be any countable open covering of X such that $G_n \subset G_{n+1}$, $n = 1, 2, \cdots$. Let us put

$$F_n = X - \mathrm{St}^2(X - G_n, \mathfrak{A}_n),\ n = 1, 2, \cdots.$$

Then $X = \cup F_n$. Indeed, if not, then there exists a point x_0 of X such that $x_0 \in X - \cup F_n = \cap \mathrm{St}^2(X - G_n, \mathfrak{A}_n)$, and hence $\mathrm{St}^2(x_0, \mathfrak{A}_n) \cap (X - G_n)$

$\neq \emptyset$ for $n=1, 2, \cdots$. This shows that $\cap(X-G_n)\neq\emptyset$ by (M_2), which is a contradiction. Hence $X=\cup F_n$. Now let us put

$$H_n = X - \overline{\mathrm{St}(X-G_n, \mathfrak{A}_n)},\ n=1, 2, \cdots.$$

Then clearly $F_n \subset H_n$ for each n, and hence $X=\cup H_n$. Further it holds that $\bar{H}_n \subset G_n$ for each n. Consequently, by a theorem of F. Ishikawa [4], X is countably paracompact. Thus we complete the proof.

Proof of Theorem 2.4. Let X be a normal wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2). As is proved by M. Katetov [5], a normal space is strongly normal if and only if for every locally finite collection $\{F_\lambda\}$ of closed subsets of X there exists a locally finite collection $\{H_\lambda\}$ of open subsets of X such that $F_\lambda \subset H_\lambda$ for each λ. To apply this theorem to our case, let $\{F_\lambda\}$ be a locally finite collection of closed subsets of X. Then it is easily proved by (M_2) that for each point x of X there exists some $\mathfrak{A}_n$ such that $\{\lambda \mid \mathrm{St}^2(x, \mathfrak{A}_n)\cap F_\lambda\neq\emptyset\}$ is a finite set. For each n, let us denote by A_n the subset of X consisting of points x of X such that $\{\lambda \mid \mathrm{St}^2(x, \mathfrak{A}_n) \cap F_\lambda\neq\emptyset\}$ is a finite set, and put $B_n = \mathrm{Int}\, A_n$. Then clearly $B_n \subset B_{n+1}$ for each n, and further it is proved that $\{B_n\}$ is an open covering of X. Indeed, let $x_0 \in X$. Then, in view of Lemma 2.5, there exists some $\mathfrak{A}_n$ such that $\{\lambda \mid \mathrm{St}^3(x_0, \mathfrak{A}_n)\cap F_\lambda\neq\emptyset\}$ is a finite set. Therefore, for each point x of $\mathrm{St}(x_0, \mathfrak{A}_n)$, $\{\lambda \mid \mathrm{St}^2(x, \mathfrak{A}_n)\cap F_\lambda\neq\emptyset\}$ is a finite set. This shows that $\mathrm{St}(x_0, \mathfrak{A}_n)\subset A_n$, i.e., $x_0 \in B_n = \mathrm{Int}\, A_n$, and hence $X=\cup B_n$. Now, since X is countably paracompact by Lemma 2.6, there exists a locally finite open refinement $\{G_n\}$ of $\{B_n\}$ such that $\bar{G}_n \subset B_n$ for each n. Let us put $G_{\lambda n} = \mathrm{St}(F_\lambda, \mathfrak{A}_n)\cap G_n$ and $H_\lambda = \bigcup_{n=1}^{\infty} G_{\lambda n}$. Then clearly $F_\lambda \subset H_\lambda$ for each λ, and further $\{H_\lambda\}$ is a locally finite collection of open subsets of X. Indeed, let $x_0 \in X$, and $U(x_0) = X - \cup\{\bar{G}_n \mid x_0 \notin \bar{G}_n\}$. Since $\{\bar{G}_n \mid n=1, 2, \cdots\}$ is locally finite in X, $U(x_0)$ is an open neighborhood of x_0. Let $\{G_{n(i)} \mid i=1, \cdots, k\}$ be all of the elements of $\{G_n\}$ each closure of which contains x_0. Then from $x_0 \in \bar{G}_{n(i)} \subset B_{n(i)}$, $i=1, \cdots, k$, it follows that $\{\lambda \mid \mathrm{St}^2(x_0, \mathfrak{A}_{n(i)})\cap F_\lambda\neq\emptyset\}$ is a finite set for $i=1, \cdots, k$. This implies that $\{\lambda \mid \mathrm{St}(x_0, \mathfrak{A}_{n(i)})\cap \mathrm{St}(F_\lambda, \mathfrak{A}_{n(i)})\neq\emptyset\}$ is a finite set for $i=1, \cdots, k$. Hence $\{\lambda \mid \mathrm{St}(x_0, \mathfrak{A}_{n(i)})\cap G_{\lambda n(i)}\neq\emptyset\}$ is also a finite set for each $i\leqq k$. Let us put $m=\mathrm{Max}\{n(1), \cdots, n(k)\}$, $V(x_0)=\mathrm{St}(x_0, \mathfrak{A}_m)\cap U(x_0)$, $\Lambda_i = \{\lambda \mid V(x_0)\, G_{\lambda n(i)}\neq\emptyset\}$ and $\Gamma = \bigcup_{i=1}^{k} \Lambda_i$. Then Λ_i is a finite set for each $i\leqq k$, and hence so is Γ. Further $V(x_0)$ intersects only elements H_λ such that $\lambda\in\Gamma$. Consequently $\{H_\lambda\}$ is locally finite in X. Thus we complete the proof.

In spite of validity of Theorem 2.4, we don't know whether every normal wM-space is an M-space or not.

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