22. Projective R-Modules with Chain Conditions on R/J

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Introduction. Let R be a ring and J its Jacobson radical.

In [6], I. Kaplansky proved the useful theorem that, if R is local, i.e. R/J is a division ring, then any projective R-module is free. In [4], Y. Hinohara generalized this theorem by proving that, if R is commutative and semilocal and its space of prime ideals spec (R) is connected, then any projective R-module is free. H. Bass, in his recent book [2], defines a noncommutative ring R to be *semilocal* if R/J is Artinian. §1 of the present paper is devoted to proving the analogue of the above results for semilocal rings. The local and commutative cases are consequences of this result (§1. Theorem).

In the case that the projective module is nonfinitely generated, our proof depends upon the following theorem of Bass [1].

Theorem. *If R/J is left Noetherian, then any uniformly $\aleph$-big projective R-module is free.*

The paper by Bass appeared as a reference in several articles. In particular the result has been useful if R is the group ring of a finite group [7, Theorem 7] and in the study of topological spaces homotopically equivalent to finite dimensional complexes [8, discussion of Theorem E].

The proof of the theorem is short and very elegant in the case that the cardinal $\aleph$ is uncountable [1, Theorem 2.2]. But the countable case is much more involved. In §2 of the present paper we give an elementary proof of Bass' result which would avoid the second half of the argument in the establishment of [1, Theorem 3.1] involving the juggling of infinite matrices. The proof is a positive response to [1, beginning of §3] and is the cleanest.

§1. Following [1], we shall call a ring R *p-connected* if for each nonzero projective R-module P, the image $\tau(P)$ of the natural pairing: $\mathrm{Hom}_R(P, R)\otimes P\to R$ is all of R. Regardless, $\tau(P)$ is a two-sided ideal. Any local ring R is p-connected. This statement can be proved without using the theorem of Kaplansky except for two simple lemmas. We need only assume that R/J be a simple ring. For, let $P\neq(0)$ be a projective R-module. It is easy to show that $P=\tau(P)P$ [see 3, p. 132]. Suppose $\tau(P)$ is a proper ideal of R. Then, since R/J is simple, $\tau(P)$

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$\subset J$ and hence $P=JP$. But by Bass' lemma [4, Lemma 3], JP is always a proper submodule of P.

Over a commutative ring R, it is apparent that R is p-connected if and only if each nonzero projective R-module P does not equal $\mathfrak{M}P$ for all maximal ideals $\mathfrak{M}$ of R. Thus, by [4, Lemma 4], a commutative semilocal ring R is p-connected if and only if spec (R) is connected.

Proposition. *Let R be a given semilocal ring. Then R is p-connected if and only if for each nonzero projective R-module P, P/JP is R/J-faithful.*

Proof. As indicated above, $P=\tau(P)P$. Hence
$\tau(P)(P/JP)=(\tau(P)P+JP)/JP=P/JP$ and so $(\tau(P)+J)P/JP=P/JP$. Now suppose P/JP is faithful over the semisimple Artinian ring R/J. Then $\tau_{R/J}(P/JP)=R/J$ and so $\tau(P)+J=R$. We conclude that $\tau(P)=R$ by Nakayama's lemma. Conversely, suppose we can write the identity of R as

$1=\sum_i h_i(p_i)$, $h_i\in \mathrm{Hom}_R(P, R)$. Let $x\in Ann_R(P/JP)$. Then, $xp_i=\sum_j \lambda_{ij}p_{ij}$, for some $\lambda_{ij}\in J$, $p_{ij}\in P$. Therefore $x=\sum_{i,j}\lambda_{ij}h_i(p_{ij})\in J$ and the proof is complete.

Theorem. *Let R be a given semilocal ring. If R is p-connected and for every maximal two-sided ideal $\mathfrak{M}$ of R, $R/\mathfrak{M}$ is a division ring, then every projective R-module is free.*

Proof. We first prove that any nonzero projective R-module P has a direct summand isomorphic to R. Since the semisimple ring R/J is Artinian, R has only a finite number of maximal two-sided ideals, say $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. We can then find elements $r_j\in R$ such that $r_j\equiv 1 \bmod \mathfrak{M}_j$ and $r_j\equiv 0 \bmod \mathfrak{M}_i$, $i\neq j$ (Chinese Remainder Theorem); in fact, for each $i\neq j$ there are $c_i\in\mathfrak{M}_i$ and $d_i\in\mathfrak{M}_j$ such that $c_i+d_i=1$ and setting $r_j=c_1\cdots c_{j-1}c_{j+1}\cdots c_n$ give us the desired elements. Since $\tau(P)=R$, there exist homomorphisms $\varphi_j\in\mathrm{Hom}_R(P, R)$ and elements $p_j\in P$ such that $\varphi_j(p_j)\notin\mathfrak{M}_j$. Let $p=\sum_{j=1}^n r_jp_j$. Then $\varphi_j(p)\not\equiv 0 \bmod \mathfrak{M}_j$. Next, define $\varphi: P\to R$ by setting $\varphi(x)=\sum_{j=1}^n \varphi_j(x)r_j$, all $x\in P$. Then $\varphi(p)\not\equiv 0 \bmod \mathfrak{M}_j$, $1\leq j\leq n$. Since each $R/\mathfrak{M}_j$ is a division ring, we can find $l_j\in R$ such that $l_j\varphi(p)\equiv 1 \bmod \mathfrak{M}_j$. Let $l=\sum_{j=1}^n r_jl_j$. Then $l\varphi(p)\equiv 1 \bmod \mathfrak{M}_j$, $1\leq j\leq n$. We conclude that $1-l\varphi(p)\in J$ and hence that $l\varphi(p)$ is a unit in R. Therefore φ is an epimorphism and P has a direct summand isomorphic to R. It is now evident that the sufficiency holds for finitely generated projective R-modules. But if the projective R-module P is not finitely generated, then the above process may be applied to P any finite number of times: $P\cong R\oplus R\oplus\cdots\oplus R\oplus S_k$ (k R-

summands). In this case we shall show that $P/\mathfrak{A}P$ is not finitely generated for any proper two-sided ideal $\mathfrak{A}$ of R. For this purpose it suffices to show that $P/\mathfrak{M}P$ is not finitely generated for any $\mathfrak{M} \in \{\mathfrak{M}_1, \cdots, \mathfrak{M}_n\}$. Suppose $P/\mathfrak{M}P$ is finitely generated. Then there exists an $R/\mathfrak{M}$-module $\bar{Q}$ such that $P/\mathfrak{M}P \oplus \bar{Q}$ is a finitely generated free $R/\mathfrak{M}$-module, say $\sum^{k} \oplus R/\mathfrak{M}$. But $P/\mathfrak{M}P \cong R/\mathfrak{M} \oplus \cdots \oplus R/\mathfrak{M} \oplus S_{k+1}/\mathfrak{M}S_{k+1}$ and therefore $\sum^{k+1} \oplus R/\mathfrak{M} \oplus S_{k+1}/\mathfrak{M}S_{k+1} \oplus \bar{Q} \cong \sum^{k} \oplus R/\mathfrak{M}$. Then, since $R/\mathfrak{M}$ is a division ring and hence certainly semilocal, this last isomorphism and cancellation [2, p. 168, Corollary 1.4] imply that $R/\mathfrak{M} \oplus S_{k+1}/\mathfrak{M}S_{k+1} \oplus \bar{Q} \cong (0)$ which is impossible. We conclude that $P/\mathfrak{A}P$ is not finitely generated for any proper ideal $\mathfrak{A}$ of R. Thus, by definition, P is uniformly big. Since R/J is Noetherian, Bass' Theorem (see Introduction) enables us to conclude that P is free. This completes the proof of the theorem.

Remarks. (a) According to the discussion at the beginning of this section, it follows that the results, stated in the Introduction, of Kaplansky and Hinohara are included in this theorem. (b) If one removes the hypothesis, $R/\mathfrak{M}$ is a division ring, then the result is no longer true; for example, consider a simple Artinian ring (hence semilocal) which is the direct sum of at least two simple (isomorphic) left ideals.

§ 2. We observe that the same elegant proof of [1, Theorem 2.2] can be used for Bass' Theorem in the countably infinite case provided the projective module can be expressed as an infinite direct sum. For this purpose, we incorporate the methods of [1] and [5].

Lemma. *Suppose R/J is left Noetherian. Let P be a uniformly $\aleph_0$-big projective R-module, u any element of P and M a submodule of P such that $Ru+M=P$. Then there exists an element $m \in M$ such that $R(u+m)$ is a free direct summand of P.*

Proof. Write $F=P\oplus Q$, F a free module whose elements are written as (ultimately zero) sequences of elements of R. If

$\beta=(b_1, b_2, \cdots) \in F$, let $S(\beta)=\{i \mid b_i \notin J\}$, $n(\beta)=\max\{i \mid b_i \neq 0\}$, $(\beta)'=$ the two-sided ideal generated by the b_i for all i, and $[\beta)=$ the right ideal generated by those $b_i \notin J$. Now let $\{\alpha_1, \alpha_2, \cdots \alpha_n, \cdots\}$ generate M and $\alpha_n=(a_{n1}, a_{n2}, \cdots, a_{nk}, \cdots)$. Since R/J is left Noetherian, we may assume that the row finite matrix $A=(a_{ij})$ $i, j=1, 2, \cdots$, is also column finite mod J. This assumption and the fact that M is not finitely generated imply that we can choose $n_1 > n(u)$ so that if $n \geq n_1$, then $a_{ni} \in J$ for $i=1, 2, \cdots, n(u)$. Thus $S(\alpha_n) \cap S(u)=\emptyset$ for all $n \geq n_1$. Set $L_1=R(\sum_{k \in S(\alpha_{n_1})} a_{n_1 k})+J$ and assume $L_1 \neq R$. By the column finiteness mod J of A, we can choose an $N > n_1$ so that if $n \geq N$, then

$S(\alpha_n)\cap S(\alpha_m)=\emptyset$ for all $m\leq n_1$. Since P is generated by $\{u, \alpha_1, \alpha_2, \cdots\}$ and P is uniformly $\aleph_0$-big we can choose $n_2\geq N$ so that $[\alpha_{n_2})\not\subseteq L_1$; for otherwise, if $\mathfrak{A}=\sum_{n\geq N}(\alpha_n)'$, then $\mathfrak{A}\neq R$ and $P/\mathfrak{A}P$ is finitely generated. Pick $l_2=\sum_{j\in S(\alpha_{n_2})} a_{n_2j}s_{2j}\notin L_1$, $s_{2j}\in R$. Then $L_2=L_1+Rl_2$ properly contains L_1. Assume $L_2\neq R$. Then, as before, we can choose $n_3>n_2$ so that if $n\geq n_3$, then $S(\alpha_n)\cap S(\alpha_m)=\emptyset$ all $m\leq n_2$, and $l_3=\sum_{j\in S(\alpha_{n_3})} a_{n_3j}s_{3j}\notin L_2$, $s_{3j}\in R$. Set $L_3=L_2+Rl_3$ and we have $L_1\subsetneqq L_2\subsetneqq L_3$. In this way we construct a sequence of left ideals $\{L_i\}$ which strictly increases mod J. Since R/J is left Noetherian, $L_k=R$ for some k. Hence we can write the identity of R in the form,

$$(*)\qquad 1=\sum_{i=1}^{k}\sum_{j\in S(\alpha_{n_i})} r_i a_{n_ij}s_{ij} \text{ for suitable } r_i,\ s_{ij}\in R.$$

Let $m=\sum_{i=1}^{k} r_i\alpha_{n_i}\in M$. Then, for each $m_n=\sum_{i=1}^{k} r_i a_{n_in}$, we have $a_{n_in}\in J$ for at most one $i\leq k$, since $S(\alpha_{n_i})\cap S(\alpha_{n_j})=\emptyset$ if $i\neq j$. Thus by $(*)$, we can write $1=\sum_{j\in S(m)} m_jc_j+d$ for suitable $c_j\in R$ and $d\in J$. Therefore $[m)=R$. But since $n(u)<j$ for all $j\in S(m)$, $[u+m)=R$. This implies that $R(u+m)$ is a free direct summand of P and the proof of the lemma is complete.

Proof of Bass' Theorem (in the countably infinite case). Because of the above lemma, we are able to duplicate the elementary iterative procedure of the proof of [5, p. 87, Lemma 7.2] in order to show that each element of P can be embedded in a finitely generated direct summand of P. Hence, according to [6, Lemma 1], P is a direct sum of finitely generated modules. The proof of [1, Theorem 2.2] can now be used to conclude that P is free.

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