67. *Characterizations of Strongly Regular Rings. II*

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An associative ring A is called strongly regular if, for any element a of A, there exists an element x in A such that $a=a^2x$. Characterizations of strongly regular rings were given by Andrunakievič [1], Lajos [5], Luh [8], and Schein [9], moreover by Lajos and Szász [7]. This paper is connected with authors' earlier note [6]. For the semigroup theoretical terminology we refer to Clifford and Preston [2].

The purpose of this note is to give a further characterization of the class of strongly regular rings. For this aim we shall use four well known important propositions and in the proof of the theorem an equivalence relation discussed formerly by S. Lajos [3], which is a two-sided congruence relation on the multiplicative semigroup S of a strongly regular ring A.

Proposition 1. *A strongly regular ring A has no non-zero nilpotent elements.*

Proof. Obviously $a=a^2x$ implies $a^3x^2=a^2x=a$ and $a^{n+1}x^n=a$ where $a\in A$ and n is an arbitrary positive integer. Therefore $a^n=0$ implies $a=0$, for any element a of A.

Proposition 2. *Any idempotent element of a strongly regular ring A lies in the center of the ring.*

Proof. For $e=e^2$ and any $x\in A$ Proposition 1 and the relation

$$(exe-xe)^2=exexe-exexe-xexe+xexe=0$$

imply $exe=xe$. Similarly we have $exe=ex$, that is $ex=xe$.

Proposition 3. *Any strongly regular ring A is regular.*

Proof. Let a be an arbitrary element of A. Then $a=a^2x$ implies

$$(a-axa)^2=0.$$

Hence by Proposition 1, $a=axa$. Therefore $e=ax$ and $f=xa$ are idempotent elements and Proposition 2 implies $a=af=fa=axa=xa^2$.

Proposition 4. *Any strongly regular ring A is a two-sided ring.*

Proof. For any element a of A, the principal right ideal $(a)_R$ of A is a two-sided ideal because $a=a^2x=axa$, $(ax)^2=ax$,

$$(a)_R=(ax)_R=(axa)_R$$

and $yax=axy$ for any element y of A, by Proposition 2. Analogously can be proved that every principal left ideal of A is also two-sided.

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Next we formulate the result characterizing the class of strongly regular rings.

Theorem 1. *An associative ring A is strongly regular if and only if its multiplicative semigroup S is a semilattice of groups.*

Proof. Let A be a strongly regular ring. We define an equivalence relation σ as follows:

$$a\sigma b \text{ if and only if } aA=bA$$

for any elements a, b of A. It is evident that σ is an equivalence relation on S, moreover it is a left congruence on S, and σ is a two-sided congruence relation on S by Proposition 4. Therefore σ induces a decomposition of S into pairwise disjoint classes S_a, where $a \in A$. We show that for any non-zero element a of A, the class S_a is a group. If $b \in S_a$ then $aA=bA$ and

$$abA=abA^2=a(bA)A=a(Ab)A=aA \cdot bA,$$

moreover $(aA)^2=aA$ imply $abA=aA$, i.e. $a\sigma ab$. Therefore S_a is a semigroup. Next we show that any class S_a is both left and right simple. If $x_1 y \in S_a$ then $y \in yA=xyA$. Thus there exists an element z of A such that $y=xyz$. Put $u=yz$. We show that $u \in S_a$. First there exists an element v in A such that $z=z^2v$. This implies

$$y=xyz=xy \cdot z^2v=(xyz)(zv)=yzv=uv,$$

that is $yA \subseteq uA$. Starting with the element $u=yz$ similarly we get $uA \subseteq yA$. Therefore $yA=uA$ and $u \in S_a$. Thus $xS_a=S_a$ for any $x \in S_a$, i.e. the class S_a is a right simple semigroup. The proof of the left simplicity of S_a is the left-right dual of the above and we omit it. We conclude that S_a is a group and the multiplicative semigroup S of A is the union of the disjoint classes S_a, every of which is a group.

We show that S is a semilattice of groups. Let e_a be the identity element of S_a. Then e_a lies in the center of A by Proposition 2, and the product $e_a e_b$ is also an idempotent element which must coincide with the identity element e_c of S_c for a suitable element c of A. Then $a\sigma e_a$ and $b\sigma e_b$ imply $ab\sigma e_c$ and $ba\sigma e_c$, where $e_a e_b=e_b e_a=e_c$. Therefore S is a semilattice of groups.

Conversely, assume that A is an associative ring and its multiplicative semigroup S is a semilattice of groups. If G_a denotes the subgroup of S containing a and x is the inverse element of a in G_a, then $ax=xa=e_a$, where e_a is the identity element of G_a. This implies

$$a=ae_a=a-ax=a^2x \qquad (a \in A)$$

showing the strong regularity of the ring A.

This completes the proof of our Theorem 1.

The following result is a consequence of Theorem 1 and other known characterizations of strongly regular rings.

Theorem 2. *For an associative ring A the following conditions* *are equivalent*:

(1) *A is strongly regular.*

(2) *The multiplicative semigroup of A is an inverse semigroup.*

(3) *The multiplicative semigroup of A is a semilattice of groups.*

(4) *A is a regular duo ring.*

Remark. The condition (V) in the first part of this paper is not complete, the correct form of it reads as follows:

(V) *A is regular and it is a subdirect sum of division rings.*

References

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