65. *On Some Boundary Properties of Harmonic Dirichlet Functions*

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Introduction. The fundamental potential functions on open Riemann surfaces like Green function, harmonic measure etc. show very specific boundary behaviors provided the surfaces have smooth boundaries. But on general surfaces the situation become complicated, actually one has first to define the boundary values or normal derivatives at the ideal boundary.

We have shown [5] that canonical potentials, especially harmonic measures assume constant values quasi-everywhere on each component of Kuramochi boundary. For other compactifications such a property was investigated afterwards by Kusunoki-Mori [6], Ikegami [4] and Watanabe [9] (cf. also Nakai-Sario [8]). At the same time it was inquired whether this boundary behavior would characterize those functions, however the question is still open. The purpose of this paper is to give some comments to this problem from the viewpoint of normal derivatives.

1. In the following we shall use some terminologies and notations without repetitions (cf. Constantinescu-Cornea [2] and Ahlfors-Sario [1]). Let R be a hyperbolic Riemann surface and R^* a resolutive compactification of R. Let ω_a be the harmonic measure of the ideal boundary $\Delta=R^*-R$ with respect to a point $a\in R$. The carrier of ω_a coincides with the harmonic boundary Δ_0 of Δ. For fixed $a=a_0$ we denote ω_{a_0} by ω, and by $L^2(\Delta)$ the Hilbert space of real-valued functions on Δ square integrable with respect to $d\omega$.

A resolutive compactification R^* is called D-normal (Maeda [7]) if every $u\in HD(R)$ (space of harmonic Dirichlet functions on R) can be expressed as

$$u(a)=\int_\Delta f\, d\omega_a=H_f(a)$$

with a resolutive function f on Δ. The compactifications of Wiener, Royden, Martin and Kuramochi are all D-normal and of type S. In the sequel we shall assume R^* is D-normal, unless otherwise stated. The mapping

$$u\rightarrow f$$

is linear and one-to-one (cf. [6]). We call f the *boundary value* (*or* *boundary function*) of u and denote it by $\tilde{u}(=H^{-1}u$ in [6]). In four important compactifications above it is known that $\tilde{u}$ actually turns out to be the limiting value of u in respective senses (cf. [6], [4]).

The following fact (cf. Doob [3], Maeda [7]) will be used later.

Lemma 1. *If $u \in HD(R)$, then $\tilde{u}$ belongs to $L^2(\Delta)$ and if $u(a_0)=0$,*

$$\|\tilde{u}\|_{\Delta}=\left|\int_{\Delta}\tilde{u}^2 d\omega\right|^{\frac{1}{2}}\leqq M\|du\|_R$$

with a positive constant M independent of u.

2. We consider a partition $\boldsymbol{P}=\{\Delta_\alpha\}$ of the ideal boundary $\Delta=\bigcup_\alpha \Delta_\alpha$ such that each Δ_α consists of mutually disjoint connected components of Δ. For the real Hilbert space $\Gamma_{he}=\{du\,;\,u\in HD(R)\}$ let

$$\Gamma_{\boldsymbol{P}}=\{du\in\Gamma_{he}\,;\,\tilde{u} \text{ is constant } \omega-\text{a.e. on each part } \Delta_\alpha \text{ of } \boldsymbol{P}\}.$$

where the exceptional sets are subsets of a set of harmonic measure zero on Δ. Evidently $\Gamma_{\boldsymbol{I}}=\{0\}$ for the identity partition $\boldsymbol{I}$. Now we have

Theorem 1. *$\Gamma_{\boldsymbol{P}}$ is a Hilbert subspace of Γ_{he}. Moreover if the compactification R^* is of type S, then $\Gamma_{\boldsymbol{P}}\supset(\boldsymbol{P})\Gamma_{hm}$, in particular $\Gamma_{\boldsymbol{Q}}\supset\Gamma_{hm}$ for the canonical partition $\boldsymbol{Q}$.*

This is seen immediately from Theorems 3, 4 (or the proofs) in [6] and it is our main concern whether $\Gamma_{\boldsymbol{P}}$ would coincide with $(\boldsymbol{P})\Gamma_{hm}$. Two spaces coincide if the partition is finite. In the following we shall assume that R^* is of type S as well as D-normal.

3. Let $\boldsymbol{P}$ be a partition of Δ and denote

$$M_{\boldsymbol{P}}=\left\{\varphi\in L^2(\Delta)\,;\int_{\Delta_\alpha}\varphi d\omega=0 \text{ for every } \Delta_\alpha \text{ in } \boldsymbol{P}\right\}.$$

It is easily seen that $M_{\boldsymbol{P}}$ is a closed subspace of $L^2(\Delta)$. Let Γ be any closed subspace of Γ_{he}, then we say that a function $u\in HD(R)$ has Γ-normal derivative $\varphi=\varphi_u$ if there exists $\varphi\in L^2(\Delta)$ such that

$$\langle dv, du\rangle_R=\int_{\Delta}\tilde{v}\varphi d\omega, \text{ for any } dv\in\Gamma$$

where $\langle dv, du\rangle_R=\int_R dv\wedge {}^*du$. Then clearly $\varphi\in M_{\boldsymbol{I}}$. Note that for dv boundary function $\tilde{v}$ is determined up to a constant, but does not matter since $\int_\Delta\varphi d\omega=0$. We shall write $N_{\boldsymbol{P}}(\Gamma)=\{du\in\Gamma\,;\ u$ has a Γ-normal derivative $\in M_{\boldsymbol{P}}\}$ and $N(\Gamma)=N_{\boldsymbol{I}}(\Gamma)$.

Lemma 2. *Let Γ be a closed subspace of Γ_{he}. Given $\varphi\in M_{\boldsymbol{P}}$, there exists $du\in N_{\boldsymbol{P}}(\Gamma)$ such that φ is the Γ-normal derivative of u. Further $N(\Gamma)$ is dense in Γ.*

Proof. Consider a linear functional on Γ:

$$dv\to\int_\Delta\tilde{v}\varphi d\omega,\qquad dv\in\Gamma$$

We may assume $v(a_0)=0$. By Lemma 1 we have then $\left|\int_\Delta\tilde{v}\varphi d\omega\right|$

$\leqq\left|\int_{\Delta}\tilde{v}^2 d\omega\int_{\Delta}\varphi^2 d\omega\right|^{\frac{1}{2}}\leqq M\|\varphi\|_{\Delta}\|dv\|_R$, hence the linear functional on Γ is bounded and by Riesz theorem there exists $du\in\Gamma$ such that $\int_{\Delta}\tilde{v}\varphi d\omega$ $=\langle dv, du\rangle_R$ for any $dv\in\Gamma$, which implies that φ is a Γ-normal derivative of u.

To show that $N(\Gamma)$ is dense in Γ suppose the contrary. Then there is $dv_0\in\Gamma$ such that $v_0(a_0)=0$, $v_0\not\equiv 0$ and $\langle dv_0, du\rangle_R=0$ for any $du\in N(\Gamma)$. While

$$\int_{\Delta}\tilde{v}_0 d\omega=v_0(a_0)=0 \quad\text{and}\quad \tilde{v}_0(\not\equiv 0)\in L^2(\Delta)$$

so $\tilde{v}_0\in M_I$ and it is a Γ-normal derivative of some u_0 with $du_0\in N(\Gamma)$, hence $0=\langle dv_0, du_0\rangle_R=\int_{\Delta}\tilde{v}_0^2 d\omega\neq 0$, which is a contradiction.

We shall denote by $B(\Delta)(\subset L^2(\Delta))$ the set of the boundary functions of $HD(R)$ and by $C(\Delta)$ the set of (bounded) continuous functions on Δ. If our (D-normal) compactification R^* is regular [7], namely if the set $B(\Delta)\cap C(\Delta)$ is dense in $C(\Delta)$ in the uniform convergence topology, then the Γ-normal derivative is uniquely determined ω-a.e. and the mapping $du\to\varphi_u$ gives an isomorphism of $N_P(\Gamma)$ onto M_P. The compactifications of Royden and Kuramochi are regular.

Lemma 3. *If R^* is regular, then the set $B(\Delta)$ is dense in $L^2(\Delta)$.*

Indeed, the set $B(\Delta)\cap C(\Delta)$ is clearly dense in $C(\Delta)$ in L^2-norm and $C(\Delta)$ is so in $L^2(\Delta)$, hence $B(\Delta)(\supset B(\Delta)\cap C(\Delta))$ is dense in $L^2(\Delta)$.

4. Now we shall show the following

Theorem 2. *$\Gamma_P=(P)\Gamma_{hm}$ if and only if for any $du\in\Gamma_P$ there exists a sequence $\{d\omega_n\}$ in $(P)\Gamma_{hm}$ such that $(\alpha)\|\tilde{\omega}_n-\tilde{u}\|_{\Delta}\to 0$ for $n\to\infty$ $(\beta)\|d\omega_n\|_R$ are uniformly bounded.*

Proof. If $\Gamma_P=(P)\Gamma_{hm}$, for given $du\in\Gamma_P$ there exist $d\omega_n\in(P)\Gamma_{hm}$ such that $\|d\omega_n-du\|_R\to 0$ and $\omega_n(a_0)=u(a_0)$. Then (α) follows from Lemma 1 and (β) is trivial.

To show the converse suppose the contrary. Then there exists $du\in\Gamma_P$ such that $du(\not\equiv 0)$ is orthogonal to $(P)\Gamma_{hm}$. For this du take a sequence $\{d\omega_n\}$ in $(P)\Gamma_{hm}$ satisfying (α) and (β). Consider any $dv\in N(\Gamma_P)$ and let $\varphi\in M_I$ be the Γ_P-normal derivative of v, then since $\tilde{\omega}_n$ converge weakly to $\tilde{u}$ in $L^2(\Delta)$ we have

$$\lim_{n\to\infty}\langle d\omega_n, dv\rangle_R=\lim_{n\to\infty}\int_{\Delta}\tilde{\omega}_n\varphi d\omega=\int_{\Delta}\tilde{u}\varphi d\omega=\langle du, dv\rangle_R.$$

While, by Lemma 2 $N(\Gamma_P)$ is dense in Γ_P, it follows under the condition (β) that $\{d\omega_n\}$ converge weakly to du in Γ_P (cf. [10]). Hence $\|du\|_R^2=\lim_{n\to\infty}\langle d\omega_n, du\rangle_R=0$ and $du\equiv 0$, which is a contradiction.

As is seen from the proof above we have the following

Corollary. *Suppose $N(\Gamma_P)=\Gamma_P$. Then $\Gamma_P=(P)\Gamma_{hm}$ if and only* *if for any* $du \in \Gamma_P$ *there exists a sequence* $\{d\omega_n\}$ *in* $(P)\Gamma_{hm}$ *with the condition* (α).

In a very special case it was shown that one can find a sequence $\{d\omega_n\}$ satisfying (α) and (β) ([9] Theorem 2).

Next we shall give a sufficient condition for $N(\Gamma_P)=\Gamma_P$.

Theorem 3. *If the set* $B(\Delta)$ *is dense and of the Baire second category in* $L^2(\Delta)$, *then every* $u \in HD(R)$ *has* Γ_{he}*-normal derivative, hence* $N(\Gamma)=\Gamma$ *for any* $\Gamma \subset \Gamma_{he}$.

Proof. Take any $du_0 \in \Gamma_{he}$, then there is a sequence $\{du_n\}$ in $N(\Gamma_{he})$ which converge strongly to du_0. Let φ_n be Γ_{he}-normal derivatives of u_n and consider the linear functionals on $L^2(\Delta)$:

$$T_n f=\int_\Delta f\varphi_n d\omega, \quad n=1, 2, \cdots.$$

As $\varphi_n \in L^2(\Delta)$ the T_n are bounded, moreover for each $f \in B(\Delta)$ $|T_n f| \leqq M_f (n=1, 2, \cdots)$, M_f being a constant dependent only on f. Because, if $f \in B(\Delta)$ then $f=\tilde{u}$ with some $du \in \Gamma_{he}$ and $\langle du, du_n\rangle_R=\int_\Delta f\varphi_n d\omega$, hence $|T_n f| \leqq \|du\|_R \|du_n\|_R \leqq M\|du\|_R$ where M is a bound of $\{\|du_n\|_R\}$. Now since $B(\Delta)$ is of the second category in $L^2(\Delta)$ we know that the norms $\|T_n\|$ are uniformly bounded by Banach theorem and the resonance theorem (cf. [10]). While, for each $f=\tilde{u} \in B(\Delta)$ the limit

$$\lim_{n\to\infty} T_n f=\lim_{n\to\infty} \langle du, du_n\rangle_R=\langle du, du_0\rangle_R$$

exists and $B(\Delta)$ is dense in $L^2(\Delta)$, hence $\lim_{n\to\infty} T_n f$ exist for all $f \in L^2(\Delta)$ and $Tf=\lim_{n\to\infty} T_n f$ is a linear functional on $L^2(\Delta)$. Therefore by the representation theorem there exists $\varphi^* \in L^2(\Delta)$ such that $Tf=\int_\Delta f\varphi^* d\omega$ for all $f \in L^2(\Delta)$. In particular for any $du \in \Gamma_{he}$ we have

$$\int_\Delta \tilde{u}\varphi^* d\omega=T\tilde{u}=\lim_{n\to\infty} T_n\tilde{u}=\langle du, du_0\rangle_R$$

This means that u_0 has a Γ_{he}-normal derivative φ^*. q.e.d.

Remark. About the assumption that $B(\Delta)$ is dense in $L^2(\Delta)$, 1) it is always satisfied if R^* is regular (Lemma 3), 2) it can be replaced by a weaker condition that $B(\Delta)$ is dense in a sphere in $L^2(\Delta)$.

5. The semi-exactness $\left(\int_\gamma {}^*ud=\int_\gamma \frac{\partial u}{\partial n} ds=0\right.$ for any dividing cycle $\left.\gamma\right)$ suggests the following

Lemma 4. *Let* $dv \in [(P)\Gamma_{hm}]^\perp=(P)\Gamma^*_{hse}$. *If* v *has a* Γ_{he}*-normal derivative* φ, *then* φ *belongs to* M_P.

Proof. First we note that the set $\left\{\alpha ; \int_{\Delta_\alpha} |\varphi| d\omega>0\right\}$ is countable, since $\varphi \in L^2(\Delta) \subset L^1(\Delta)$. So let $\{\Delta_n\}$ be a countable number of parts of P on which the integral is positive. For each Δ_n let $\varphi_n=\varphi\chi_{\Delta_n} \in L^1(\Delta)$,

χ_{Δ_n} being the characteristic function of Δ_n. Take a decreasing sequence of non-compact subregions $G_n^\nu(\nu=1,2,\cdots)$ in R which determine Δ_n and let

$$\Delta_n^\nu=\{\Delta_\alpha\,;\,\Delta_\alpha\subset\bar{G}_n^\nu \text{ (closure in } R^*)\}.$$

Then for each ν we have a harmonic measure $d\omega_n^\nu\in(P)\Gamma_{hm}$ such that $\omega_n^\nu=1$ a.e. on Δ_n^ν and $=0$ a.e. on $\Delta-\Delta_n^\nu$. Hence $\int_\Delta\omega_n^\nu\varphi d\omega=\langle d\omega_n^\nu,dv\rangle_R$ $=0$. while $\omega_n^\nu\varphi\to\varphi_n$ a.e. $(\nu\to\infty)$ it follows by the dominated convergence theorem that

$$0=\lim_{\nu\to\infty}\int_\Delta\omega_n^\nu\varphi d\omega=\int_\Delta\varphi_n d\omega=\int_{\Delta_n}\varphi d\omega$$

which implies that $\varphi\in M_P$.

Now we shall consider a restriction to the ideal boundary $\Delta=\bigcup_\alpha\Delta_\alpha$, namely assume that the harmonic measure is decomposable:

(*) if $E_\alpha\subset\Delta_\alpha$ and $\omega(E_\alpha)=0$, then $\omega(\bigcup_\alpha E_\alpha)=0$.

For example, it is satisfied if Δ consists of a countable number of parts $\{\Delta_n\}$.

Theorem 4. *If R^* is regular and satisfies the condition (*), then it occurs that $\Gamma_P=(P)\Gamma_{hm}$ or the set $B(\Delta)$ is of the first category in $L^2(\Delta)$.*

Proof. Let $dv\in\Gamma_P$ be orthogonal to $(P)\Gamma_{hm}$. Suppose that v has a Γ_{he}-normal derivative φ, then it follows by Lemma 4 and (*) that $\varphi\in M_P$ and $\varphi=0$ ω-a.e. on $\Delta-\bigcup_{n=1}^\infty\Delta_n$. The $\tilde{v}$ is constant ω-a.e. on each Δ_n and

$$\|dv\|_R^2=\int_\Delta\tilde{v}\varphi d\omega=\sum_{n=1}^\infty\tilde{v}\int_{\Delta_n}\varphi d\omega=0$$

i.e. $dv\equiv0$. By Theorem 3 this completes the proof.

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